NOTICE TO OUR CONTRIBUTORS

We regret to announce that the policy of providing authors with 50 free reprints of their articles is being discontinued beginning with the January 1975 issue. All reprints of articles appearing in volume 48 and future volumes will have to be purchased.

THE GAME OF SIM: A WINNING STRATEGY FOR THE SECOND PLAYER

ERNEST MEAD, ALEXANDER ROSA, CHARLOTTE HUANG,
McMaster University, Hamilton, Ontario

The game of SIM [**2**, **4**] is played by two players on the six vertices of the complete graph K_6: the first and second player color alternately the 15 edges of K_6 with black and red respectively. A player wins if he forces his opponent to complete a monochromatic triangle. It is well known [**3**] that the Ramsey number $r(3,3)$ equals 6 (i.e., six is the smallest integer n such that in any coloring of the edges of the complete graph K_n on n vertices by two colors, there must be a monochromatic triangle); therefore a tie is impossible. It is also well known (see, e.g., [**1**]) that then one of the players must possess a winning strategy. It turns out that it is the second player who can be assured of his win. We were able to devise a strategy for the second player and to show that it is a winning strategy; however, a simpler (in terms of the rules to be followed) winning strategy is still desirable.

A few basic observations. The first player (i.e., the player who makes the first move) uses black and the second player uses red. The vertex-set and the edge-set of a graph G are denoted by $V(G)$ and $E(G)$ respectively; it will always be clear from the content whether G is undirected or directed. The elements of $E(K_6)$ are (unordered) pairs (i,j), $i \neq j$. We assume that in any game played, a player will never make a move that completes a triangle of his own color unless he is forced to do so. In the latter case, the game stops immediately before the losing move is made.

A *position* of a particular game after k moves is a subgraph of K_6 with k edges, each of which is colored black or red. Evidently, we must have $k \leqq 14$; let us admit also $k = 0$ and call the corresponding position *initial.* On the other hand, a position is *terminal* if the player who has to make the next move, must complete a monochromatic triangle (and therefore loses). A terminal position is said to be a *black terminal* or a *red terminal* according to whether the loser is the second player or the first player, respectively.

One could conceivably determine the winning strategy for one of the players in the game of SIM as follows:

Form a directed graph G whose vertices are all possible positions of all possible games, and whose edges are the moves, i.e., $(A,B) \in E(G)$ if there exists a particular game in which position B arises from position A after a single move. In the graph G just described, one looks for a set $S \subseteq V(G)$ with the properties:

(1) For every vertex $X \in V(G)$, $X \notin S$ such that $(Y, X) \in E(G)$ for some $Y \in S$, there is a vertex $T \in S$ such that $(X, T) \in E(G)$.

(2) $Y, \ T \in S \Rightarrow (Y, T) \notin E(G)$.

(3) Either the initial position or the position following the initial position is in S, and S includes some of the terminal positions.

The set S is a kernel of a certain subgraph of G, namely of the subgraph spanned by the vertices of S and all the successors of the vertices of S in G. Since S can obviously contain terminal positions of only one color, it determines a winning strategy for the player with the color of the terminal vertices in S.

However, although this consideration will prove useful, it probably cannot be used for actual determination of the winning strategy in the game of SIM. The reason for this is that to construct the graph G means, in effect, to perform a complete analysis of the game. Although it is evidently sufficient to take as vertices of the graph G the isomorphism classes of positions (considered as graphs), the number of possibilities (and of essentially different games of SIM which can be played) is probably large enough to prevent anyone from determining a winning strategy by using the described approach.

Our approach. However, we do not abandon the described approach altogether. We proceed as follows: we devise a set R_0 of heuristic rules to be followed by the second player. Now if one forms a graph G_0 in the same way as one formed the graph G except that the vertices of G_0 are all possible positions of all possible games in which the second player *follows the rules of* R_0, then obviously G_0 is a subgraph of G. Let S_0 be the set of vertices of G_0 which consists of all positions obtained after moves of the second player. Now one has to check whether S_0 has the properties (1), (2), (3); that is verify whether for every vertex X representing a position after a move of the first player, there is a vertex T in S_0 that can be obtained from X by a single move of the second player. If yes, then S_0 determines a winning strategy for the second player. If not, then one modifies the set of rules R_0 to obtain a new set of rules R_1; then one forms the graph G_1 and the set of vertices S_1, etc. One hopes that eventually one will arrive at a set of rules $R_i = R^*$ such that the set of vertices S_i of the corresponding graph G_i will have properties (1), (2), (3). Obviously, whether this will occur depends on how one changes the set of rules R_j to obtain R_{j+1}. One feels that R_{j+1} has to be, in a certain sense, a "refinement" of R_j, so as to take care of possibilities not covered by R_j.

Of course, this is a very loose description of a procedure which has been used by us and which eventually led to a determination of a winning strategy R^* for the second player. In the next section we describe in a more precise manner this winning strategy.

Let us call the edges of the complement of any position *free edges*. Suppose a, b, c are three vertices of K_6 such that precisely one of the three edges (a, b), (a, c), (b, c) is colored red and the remaining two edges are free. Then coloring one

one of those two edges red is said to *create a loser* (since then the third edge cannot be colored red and it becomes a losing move for the second player). If precisely one of the edges is colored black and the other two are free, then coloring either of these edges red is said to *create a partial mixed triangle.* If two of the three edges are colored (not both red) and the third edge is free, then coloring that edge red is said to constitute a *completion of a mixed triangle.*

We started with three heuristic rules considered in a hierarchy which can be formulated as follows: the second player should consider as possible moves only those free edges which when colored red will not complete a mixed triangle with two black edges (that is, unless he has no other choice) and from among these he should color red in his next move that one which, when becoming red, will

(1) create a minimum possible number of losers,

(2) complete a maximum possible number of mixed triangles,

(3) create a maximum possible number of partial mixed triangles.

Here the hierarchy of the rules is to be understood in the following sense: first one uses the rule (1) to single out those free edges which satisfy it, and only then the rule (2) is used to distinguish between free edges satisfying (1), etc. One could consider these three rules as a strategy R_0 mentioned above. Although this strategy cannot guarantee a win for the second player, it turned out to be a reasonably good approximation of what proved to be a winning strategy for the second player. We now skip the intermediate stages of the procedure described at the beginning of this section, and proceed to a description of the winning strategy.

Elements of the winning strategy. First of all, there is a need for a finer partition of $E(K_6)$ than the one given by

$$E(K_6) = A_i \cup B_i \cup N_i$$

where A_i, B_i and N_i, respectively, are the sets of black, red and free edges, respectively, after the ith move of a particular fixed game ($i = 0, 1, \cdots, t$; $t \leqq 14$).

A free edge (a, b) is said to be a *loser* for the first player (the second player, respectively) if there is a vertex v (different from a, b) such that both edges (a, v), (b, v) are colored black (red, respectively).

Now, we shall partition the set N_i as follows:

$$N_i = C_i \cup D_i \cup E_i \cup F_i$$

(some of the sets C_i, D_i, E_i, F_i may be empty), where the elements of C_i are losers for the first player, the elements of D_i are losers for the second player, the elements of E_i are losers for both players and the elements of F_i are the remaining free edges, called neutral edges.

Thus after the ith move, $i = 0, 1, \cdots, t$, $t \leqq 14$, the set $E(K_6)$ is partitioned by

$$E(K_6) = A_i \cup B_i \cup C_i \cup D_i \cup E_i \cup F_i$$

(some of the sets may be empty; for instance, $F_0 = E(K_6)$, $A_0 = B_0 = C_0 = D_0 = E_0 = \varnothing$).

When playing the game it is convenient to denote a loser for the first player by a dotted-red line with the implication that this edge can be colored red by the second player whenever he wishes. Similarly it is convenient to denote the losers for the second player and the losers for both players by dotted-black and dotted-red-black lines, respectively. Henceforth assume that all such lines are drawn as soon as the first player has completed a move and before the second player starts the decision process for his move. Also note that, unless specified otherwise, a line described as red can be solid or dotted-red and that a line described as black can be solid, dotted-black or dotted-red-black.

Assume that the second player is considering the edge (a,c) for a move (where (a,c) can be a neutral or dotted-red edge). If, with respect to any other vertex b, the two edges (a,b) and (b,c) are such that

(a) Both are red, then we say that he will *ruin a safe move* if he colors (a,c). (If one of the edges (a,b), (b,c) is dotted-red and the base of another red triangle, then we say that he will ruin a *hypothetically* safe move, otherwise a *valid* safe move.)

(b) One is red and the other neutral, then we say that he will *create a loser* (a *hypothetical* loser if the red edge is dotted, otherwise a *valid* loser).

(c) Both are colored (but not both red), then we say that he will *complete a mixed triangle.*

(d) One is black (including dotted-red-black) and one is neutral, then we say that he will *create a partial mixed triangle.*

A winning strategy for the second player is:

Rule 1. For the second move (i.e., when answering the first move of the first player), the second player should color red an edge which has no common *vertex* with the edge chosen by the first player.

Rule 2. For any move other than the second when there is at least one neutral edge, consider only these neutral edges and apply the following rules in a hierarchy in the same sense as described in the previous section:

(1) Ruin a minimum number of valid safe moves.
(2) Create a minimum number of losers (valid and hypothetical).
(3) Ruin a minimum number of hypothetically safe moves.
(4) Complete a maximum number of mixed triangles.
(5) Create a maximum number of partial mixed triangles.
(6) Create a minimum number of valid losers.

Then color any one of the edges satisfying the above rules.

Rule 3. For any move other than the second when there are no neutral edges, consider only the dotted-red edges and apply the same steps as in Rule 2.

Thus at each stage in the play of the game the free edges each act as the base of four triangles. It is a comparison of the colorings of these four triangles for each edge that determines which edge should be colored. Although the steps used in applying the strategy may seem numerous, with very little practice they are easy to use when the game is actually being played.

It turns out that the edge to be chosen as the next move by the second player is determined by the rule 2 uniquely up to an isomorphism. Notice that the second move (i.e., the first play to be made by the second player, rule 1) is in fact an exception denying the rule 2! If the second player would make that move by rule 2 and continue with rules 2 and 3, he could lose! We do not have the slightest idea why it is so.

Unfortunately, we cannot present a short and/or elegant proof of the fact that our strategy is a winning one. Most likely, in view of the complexity of the strategy, such a proof does not exist. We performed our proof in an exhaustive manner by playing all possible games in which the second player uses our strategy, in two ways: by hand, and by using a computer. When making the verification by hand, we were able to make substantial reductions by elimination of isomorphic positions. When making the verification on a computer (a CDC6400), we did not provide for a reduction of isomorphic positions, so that in several cases isomorphic games were played by the computer. For the third move (which is made by the first player) there are four essentially different possibilities, each one followed, of course, by a unique fourth move made by the second player. These first four completed moves were included into the input data and thus the program was run four times, each with different input data (see Figure 1 and Table 1).

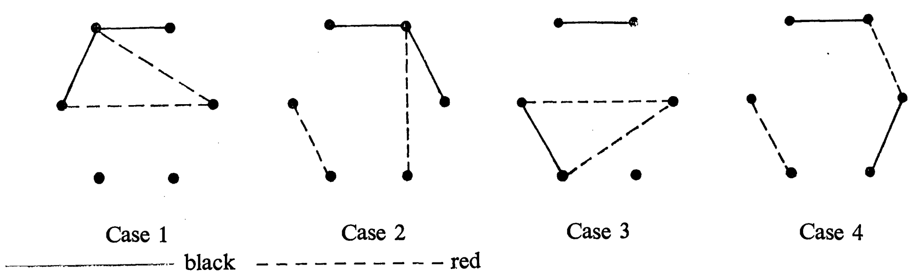

FIG. 1

TABLE 1

Case	Computer time (in sec)	Number of games played
1	73.820	509
2	75.812	644
3	80.918	569
4	96.626	742
Total	327.176	2464

References

1. C. Berge, Théorie des Graphes et ses Applications, Dunod, Paris, 1958 [English translation: Theory of Graphs, Methuen, London, 1962].

2. A. P. DeLoach, Some investigations into the game of SIM, J. Recreat. Math., 4 (1971) 36–41.

3. F. Harary, Graph Theory, Addison-Wesley, Reading, Mass., 1969.

4. G. J. Simmons, The game of SIM, J. Recreat. Math., 2 (1969) 66.