Math/CCS 103	Professor: Padraic Bartlett

Homework 15: Period Three Implies Chaos; Unit Distance Graph

Due Friday, week 8	*UCSB 2014*

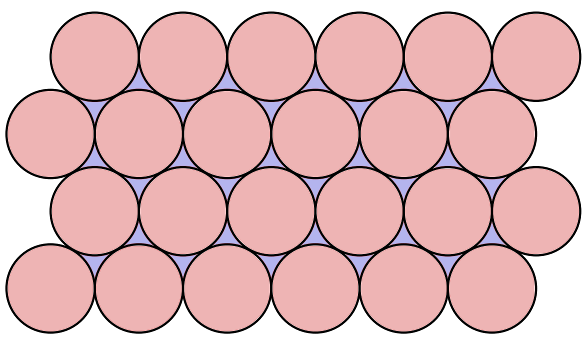
Do **two** of the **four** problems below! These may be a bit harder than earlier problems, in exchange for having only one problem set this week. Expect to spend as much time here as you would spend on two normal problem sets.

1. Find a function with a point of period 5, but no point of period 3. Prove your claim.

2. Find a function with a point of every even order, but no point of odd order. Or prove such a thing cannot exist.

3. Take Alice's construction from class, which colored the plane as follows:

   - Pack the plane with circles of diameter slightly less than one in the "optimal" fashion drawn below. Call the collection of all of these circles $\{C_i\}_{i \in I}$, for some indexing set I.
   - This packing leaves behind a collection of little Y-shapes between these circles, that also have diameter less than 1. Call the collection of these little Y-shapes $\{Y_j\}_{j \in J}$, for some indexing set J.
   - Assign each point in the plane to the shape Y_i or C_j that it is contained within. If there are multiple shapes that contain a given point in the plane, simply pick one of the possible sets that contains that point and assign that set to that point.



   We said that we can consider this assignment as a $\{C_i\}_{i \in I} \cup \{Y_j\}_{j \in J}$-coloring of the plane, in which we have assigned each point a "color", where these colors are the labels C_i, Y_j.

   (a) Explain why this is a coloring that insures that no two points with the same color are distance 1 apart.

   (b) Prove that the sets I, J are both countably infinite: i.e. that $|I| = |J| = |\mathbb{N}|$. (Do this by finding an injective map from some set you know to be countable, like $\mathbb{N}$ or $\mathbb{Q}$, to each of these sets. Or some other method that you have for demonstrating countability! Your call.)

(c) Fun fact: you don't actually need all $\mathbb{N}$ colors! I.e. when we're coloring our plane, all we need to do is keep points that are distance 1 apart different colors: so we don't necessarily need a new color for the circle that is 40 units away from the circle at the origin! I.e. we can **reuse** colors.

Find a way to "reuse" colors, so that you can actually color the plane with a **finite** number of colors so that no points distance 1 apart get the same color!

(d) What is the smallest number of colors you need?

4. Find G such that

- G is a **unit distance graph**: i.e. we can draw G in the plane with all of its edges given by straight line segments,
- $\chi(G) = 4$, and
- The chromatic number of any graph on fewer edges than G is 3.