Bertrand's Paradox!

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Let's find out.

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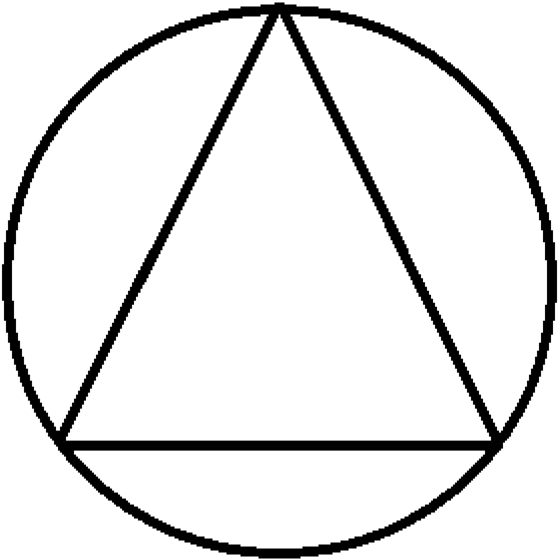
We are given a circle with an equilateral triangle inscribed in it, and asked to draw a chord through the circle randomly.

- *What is the probability that a random chord drawn through the circle is longer than the length of a side of the triangle?*



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- These methods are as follows:

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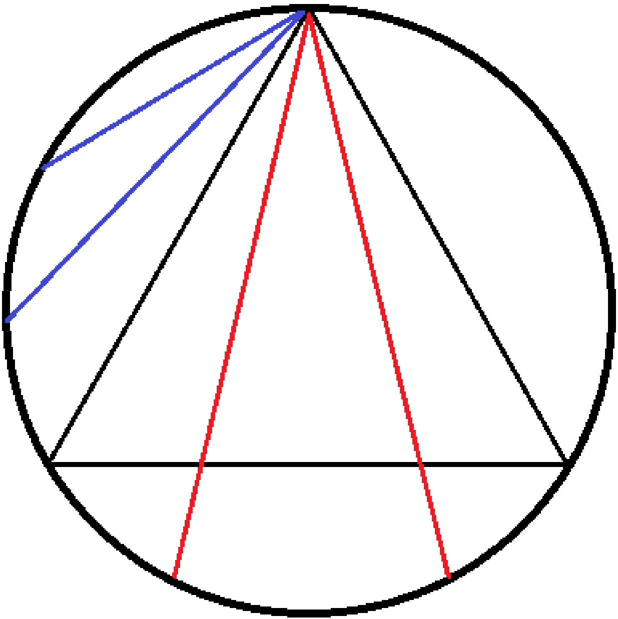
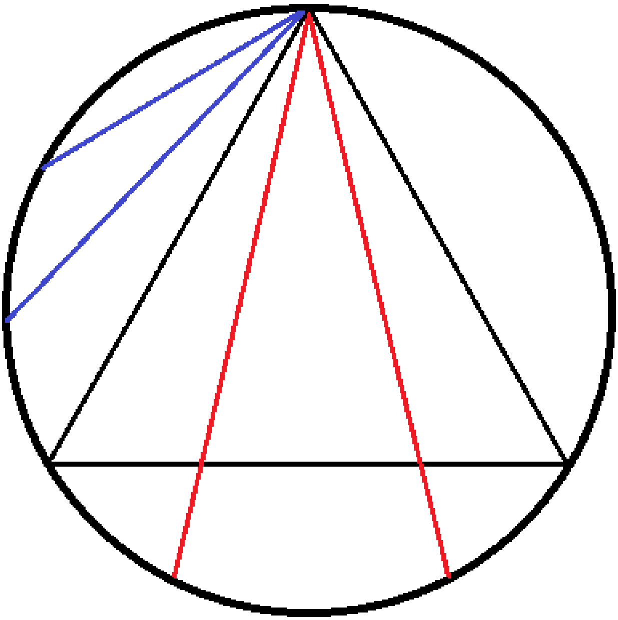
- After a chord is drawn randomly on the circle, imagine we rotate the chord so that one of the endpoints of the chord is on the chosen vertex of the triangle.
- What is the probability that any random chord has a length greater than that of one side of the equilateral triangle?





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This is a completley correct answer.

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- To do this, let's double check it with another method that Bertrand proposed.

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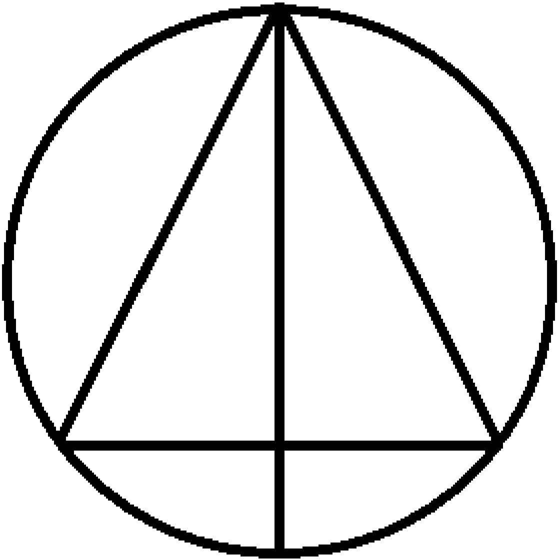
Second Method Consider a diameter of the circle drawn such that it bisects one side of the equilateral triangle.



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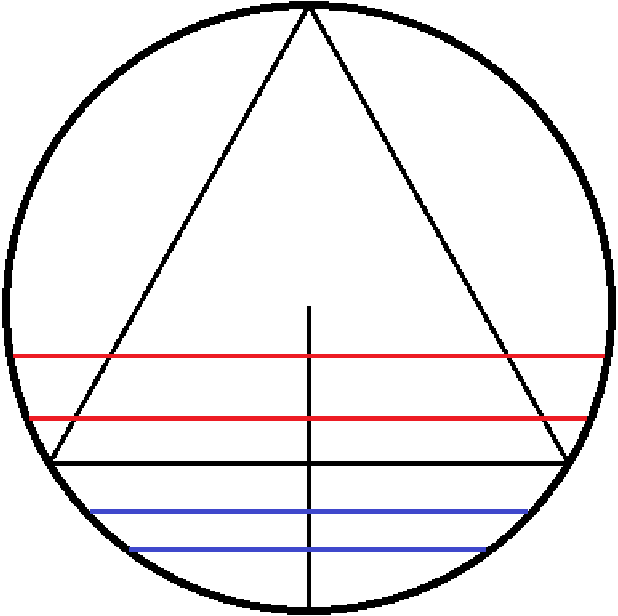
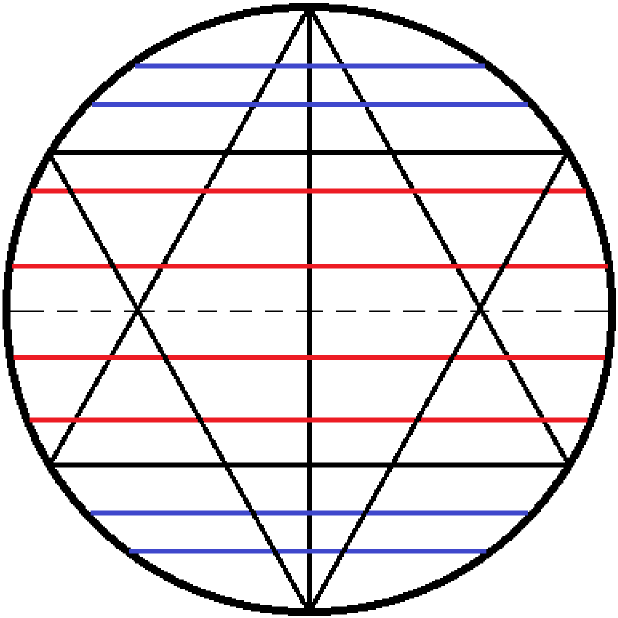
- Notice that the chords will be longer than one side of the equilateral triangle if they are between the horizontal side of the triangle and the middle of the circle.
- This is also true for the upper half of the circle, symmetrically, if we did not consider all rotations to go to the bottom.





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- The third method that Bertrand proposed must give us a solution to this paradox.

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- If the chord is longer than one side of the equilateral triangle, then its midpoint should be within a circle of radius one half the radius of the larger cicle.
- With this construction, each chord will have its own respective midpoint, except for diameters, which we know are longer than the length of a side of the triangle. This seems more fair because now most chords with the same length are accounted for, rather than being considered one chord.

- Thus, the probability that a random chord is longer than the side of the equilateral triangle inscribed in the circle is the ratio of the area of the smaller triangle to that of the big triangle.
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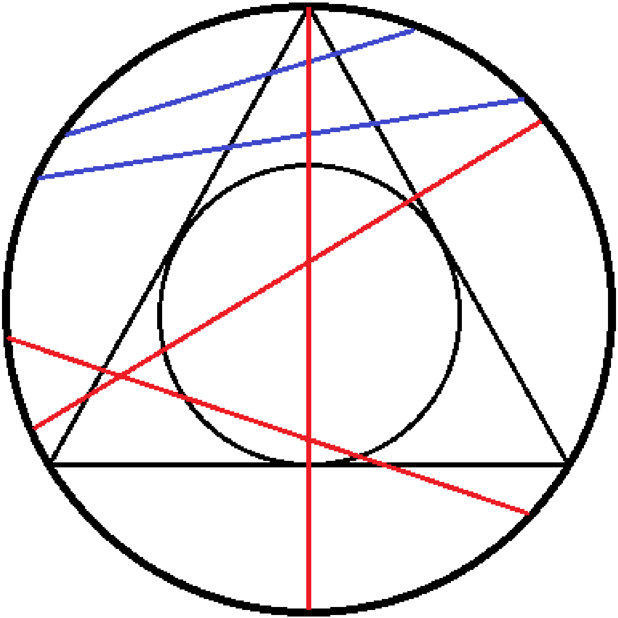
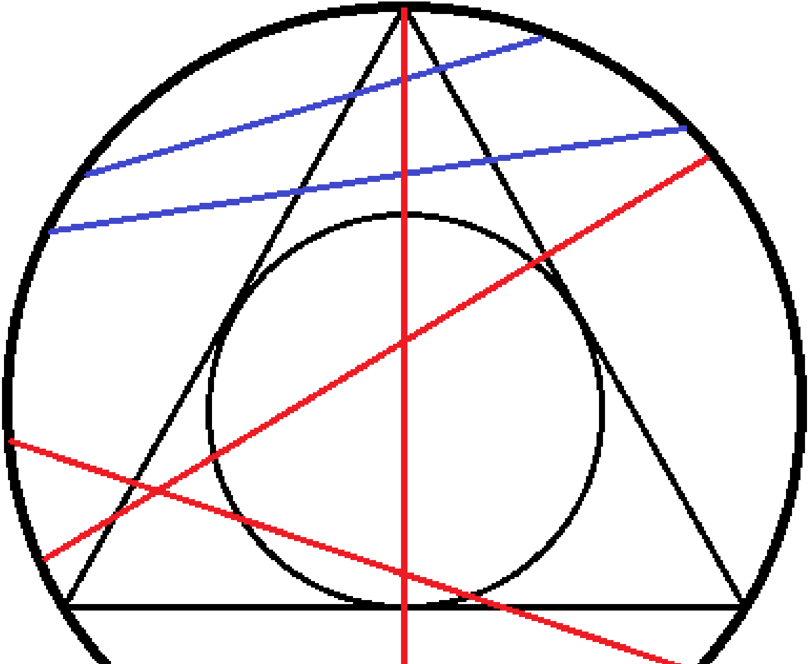
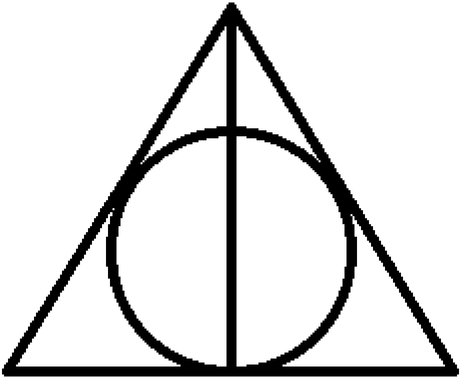


Wait, let's zoom in on that a little bit.



That looks kind of familiar, doesn't it?



HARRY!?!

Nah I'm just kidding, they aren't related at all.

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- So what do we conclude?

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- $1/3$

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- $1/3$
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- Thus, there is no solution, but merely a conclusion; Bertrand was asking for many different ways of solving the problem.
- **Especially in probability, be sure to specify exactly what the problem is asking, so that there can be no more than one solution.**

References

Here is a website that goes very in depth about solutions and helped me to write this presentation:

- http://joelvelasco.net/teaching/3865/marinoff%2094%20-%20a%20resolution%20of%20bertrand's%20paradox.pdf

Problem!

1. If we were instead given a circle with a square inside of it, what would be three different ways to check the probability that a randomly drawn chord is longer than a side of the square?