# Godel's First Incompleteness Theorem
# Lecture Notes

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### Can we prove all mathematical truths?

Can we? Many mathematicians thought we could. A major goal was to find the **axioms of mathematics**. These would be a set of axioms that give rise to a **theory** from which all mathematics would follow.

What would be a good theory of mathematics? Ideally it would be

**Consistent:** The theory does not contradict itself. That is, we cannot use our axioms to derive "$A$" and "$\neg A$."

**Complete:** From the axioms, we can derive all mathematical truths.

**Non-Trivial:** We want the theory to be useful. That is

1. We don't have something like "Every true statement is an axiom!"
2. The theory should be powerful. At least as powerful as arithmetic. (ex. we don't want the only axiom to be "0 is a number.")

Large amounts of research went into finding such a theory. Then in 1931 Kurt Gödel published his first incompleteness theorem and overturned everything. The theorem said, in essence, that we can have only two of the three properties.

## Godel's Proof

The genius of Gödel was he found a way of encoding statements into numbers. This gives each statement a unique **Gödel Number**.

If done correctly, questions about these statements – such as whether they are true or false – amounts to deciding whether their Gödel numbers have certain properties.

This allowed Gödel to "ask" a a theory $T$ questions. That is, he could input a statement into $T$. This statement will be either true or false in $T$.

> Gödel: $1 + 1 = 2$
> $T$: True
>
> Gödel: $1 + 1 = 42$
> $T$: False
>
> Gödel: "$1 + 1 = 2$" is provable in theory $T$.
> $T$: True

All is well. But then Gödel inputs the statement $G$.

> Gödel: $G$: $G$ is not provable in the theory $T$.

What happens now? If $G$ is false, then $G$ is true and we reach a contradiction. The only other potion is if $G$ is True – but then there is a statement in $T$ that $T$ cannot prove!

Either we have that $T$ is incomplete (there is something it cannot prove, namely $G$) or that $T$ is inconsistent (we have both $G$ and $\neg G$)

In a way, this is a generalization of the lair's paradox:

> "True or False: This statement is false."

In either case we reach a contradiction. The difference is that $G$ deals with provability, not truth.

One might say "Hold on! Why don't we just add $G$ into the list of axioms to make a new theory $T'$?" The problem is that $T'$ has the same flaws as $T$.

> Gödel: $G'$: $G'$ is not provable in theory $T'$.

This means that, no matter how great the axioms are, there will always unprovable statements. This is Gödel's first incompleteness theorem. In it's full (translated and paraphrased) glory, it reads

> **Gödel's First Incompleteness Theorem**
> Any axiomatic theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory

We call such statements **undecidable** (though unprovable is a more accurate word.)

## Example

Right now, it seems we can only create undecidable statements via self-reference. But we can do more. The Continium Hypothesis is a prime example. The hypothesis says

> "There is no set whose cardinality is strictly between that of the integers and the real numbers."

In 1963 Paul Cohen proved that this is undecidable, at least under the standard ZFC set theory axioms. The method he used is flexible – with it one can prove hundreds of other statements to be undecidable.

## Implications

Gödel has shown that not all truths are provable – at least from within a formal system. Truth appears to be stronger than provability.

Many questions arise from Gödel's theorems, some of the major ones being:

- If Truth $\neq$ Provable, is Reality $>$ Knowledge?
- Can we ever understand ourselves? If the mind is nothing but an overglorified computer, Gödel says we can't.

- Can we ever understand all of mathematics?

The last question comes with a free famous(ish) quotation from Gödel

> "Either mathematics is too big for the human mind or the human mind is more than a machine."

### Disclaimer

Exactly what Gödel proved is complex, and very super extra highly hyper-technical. What we have given is a layman's explanation – everything we said has caveats.

As some guy on the internet warns:

> *"The problem with Gödel's incompleteness [theorem] is that it is so open for exploitations and problems once you don't do it completely right. You can prove and disprove the existence of god... as well the correctness of religion and its incorrectness against the correctness of science. The number of horrible arguments carried out in the name of Gödel's incompleteness theorem is so large that we can't even count them all."*

### Bonus: Gödel's Second Incompleteness Theorem

We're not covering this one, but it is cool neverless.

> **Gödel's Second Incompleteness Theorem**
>
> For any axiomatic theory T including basic arithmetical truths and also certain truths about formal provability, if T includes a statement of its own consistency then T is inconsistent.

That is, $T$ cannot prove its own consistency. Poor $T$. :(