Analytic Extension of Tetration Through the Product Power-Tower

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- Tetration is iterated application of exponentiation by a:

$$tetr(a,b) = \underbrace{pow(a,pow(a,}_{b} \ldots 1 \ldots))$$

Notation & Examples

Notes on notation:

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2^x is continous on all x as well, although it's less obvious how to calculate – for instance – 2^π.
Exponentiation for positive bases can be resolved as follows:

$$(a^m)^n = a^{nm}$$

$$(a^{p/q})^q = a^p$$

So that by approximating any real power with a rational number p/q, we can define it to be the unique positive real number that, when raised to the integral power q, yields a^p.

The problem

Unfortunately, tetration has no such lovely properties, because exponentiation (unlike multiplication) does not commute:

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In particular, the inverse of any power law can be easily derived:

$$f(x) = x^{a} \qquad f^{-1}(x) = x^{1/a}$$

But no such relation holds for tetration.

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- Most functions that we deal with are analytic: polynomials, exp(x), log(x), sin(x), gamma function, Riemann zeta function...
- Anything that can be represented by a power series everywhere in the power series' area of convergence.

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We will present a method for representing tetration with a base of e as an infinite power series, in an almost entirely closed form.

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This function, like tetration, we can initially only define on the integers. Although its behavior over varying x is mostly unrelated to the behavior of tetration, we will provide an relation allowing us to compute one from the other. We will see that if we can generalize p to arbitary n, we can generalize tetration, as well.

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If we set $x = F$, then the power towers "collapse":

$$p(1, F) = e^F = F$$

$$p(2, F) = e^{e^F} e^F = e^F F = F^2$$

$$p(3, F) = e^{e^{e^F}} e^{e^F} e^F = e^{e^F} e^F F = e^F F^2 = F^3$$

Relation 1

If we take the ratio of $p(n,x)$ and $p(n-1,x)$, then we can observe that all the factors except the highest power tower drop out:

$$\frac{p(n,x)}{p(n-1,x)} = \frac{e^{e^{e^x}} e^{e^x} e^x}{e^{e^x} e^x} = e^{e^{e^x}}$$

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We have this power tower, and now we can examine its derivative:

$$\frac{d(e^{e^{e^x}})}{dx} = e^{e^{e^x}} \frac{d(e^{e^x})}{dx} = e^{e^{e^x}} e^{e^x} \frac{d(e^x)}{dx} = e^{e^{e^x}} e^{e^x} e^x = p(n,x)$$

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So we can see the derivative of the ratio of two p with different n, is actually the same function p again! Now we can also note that $\frac{p(n,1)}{p(n-1,1)} = e^{e^{e^{e^1}}} = {}^n e$.

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If we have n towers of $e^{\cdot^{\cdot^{\cdot^{F}}}}$, which each collapse down to F, we can see that $p(n, F) = F^n$. This is a statement that will simply generalize to any fractional n. Thus we have determined $p(n, x)$ for all n, at a *certain* value of x.

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So the question then is, how do we find the derivatives of p in general? We will need some other relations about in order to tell us – preferrably one involving its derivative. We just found one, however, it's not enough to provide a full solution.

Relation 2

The previous relation involved $p(n, x)$ as well as $p(n-1, x)$ – however, for our Taylor series, we would like to hold n completely constant. Thus we need some way to turn our $p(n-1, x)$ into $p(n, x)$ again.

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$$\frac{d\left(\frac{xp(n,x)}{p(n,\ln x)}\right)}{dx} = p(n,x)$$

By expanding the left hand side by rules of differentiation, we can arrive at an equation that gives us $p'(n,x)$ in terms of $p(n,x)$ alone, in particular using the fact that at our chosen $x = F$, we know that $F = \ln F$, $p(n,F) = p(n,\ln F)$, and $p'(n,F) = p'(n,\ln F)$. By taking further derivatives of each side of the equation, we can calculate more and more derivatives.

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Some derivatives

(Omitting n)

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$$p'(n, F) = \frac{F^n(F^n - 1)}{F - 1}$$

$$p''(n, F) = \frac{F^n(F^{1+2n} + 2F^{2n} - 3F^{1+n} - 3F^n + 2F + 1)}{(x^2 - 1)(x - 1)}$$

General form

$$\frac{\partial^k P(n,x)}{\partial x^k} = \frac{F^n \left(\sum_{i=0}^{k} \sum_{j=0}^{(k^2-k)/2} (-1)^{k-i} a_{i,j,k} F^{j+in} \right)}{\prod_{i=1}^{k} (F^k - 1)}$$

where $a_{i,j,k}$ are some integral constants, all positive and non-zero. Some relations have been found defining many of them and placing strong restrictions on the values they can take, but we don't yet have a formula for them.

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where $a_{i,j,k}$ are some integral constants, all positive and non-zero. Some relations have been found defining many of them and placing strong restrictions on the values they can take, but we don't yet have a formula for them. Putting it all together:

$$p(n,1) = \sum_{k=0}^{\infty} \frac{\partial^k P(n,x)}{\partial x^k} \frac{(1-F)^k}{k!}$$

$${}^{n}e = \frac{p(n,1)}{p(n-1,1)}$$

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It converges quickly one the range of $n = 0$ to $n = 1$, which is *all we need* – by raising e to that power, we can effectively add 1 to our heigh; we can calculate $^{1.5}e$ as $e^{(^{0.5}e)}$.