Math/CS 103	Professor: Padraic Bartlett

Minilecture 4: Mutually Orthogonal Latin Squares

Week 4 *UCSB 2014*

On last Friday's problem set, your first question asked you to solve the following puzzle:

Question. Take a deck of playing cards, and remove the 16 aces, kings, queens, and jacks from the deck. Can you arrange these cards into a 4×4 array, so that in each column and row, no two cards share the same suit or same face value?

The second problem asked you to connect your answer above to the concept of Latin squares! Specifically, we defined the following concept, of **mutually orthogonal Latin squares**:

Definition. A pair of $n \times n$ Latin squares are called **orthogonal** if when we superimpose them (i.e. place one on top of the other), each of the possible n^2 ordered pairs of symbols occur exactly once.

A collection of k $n \times n$ Latin squares is called **mutually orthogonal** if every pair of Latin squares in our collection is orthogonal.

Example. The grid of playing cards you constructed earlier if you answered our first question is a pair of 4×4 squares, for the reasons we discussed earlier. To further illustrate the idea, we present a pair of orthogonal 3×3 Latin squares:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} (1,1) & (2,2) & (3,3) \\ (2,3) & (3,1) & (1,2) \\ (3,2) & (1,3) & (2,1) \end{bmatrix}$$

Whenever we introduce a mathematical concept in combinatorics, our first instinct should be to attempt to count it! In other words: given an order n, what is the largest collection of mutually orthogonal Latin squares we can find? An upper bound is not too hard to find:

Proposition. For any n, the maximum number of squares in a collection of $n \times n$ mutually orthogonal Latin squares is $n - 1$.

This proposition is the first problem you're proving on this problem set!

We already know that sometimes $n - 1$ is attainable: in our example above, we found 2 orthogonal Latin squares of order 3. When can we attain this bound?

On the last HW problem of last week, you showed that for any prime p, there are at least 2 mutually orthogonal Latin squares. Your second problem for this class is to strengthen this result:

Proposition. Let $p \neq 2$ be a prime number. Then there is a collection of $p - 1$ mutually orthogonal $p \times p$ Latin squares.

This raises a natural question: what happens for nonprime powers? The answers here are weirder: in general, we do not know what the size of the largest set of $n \times n$ mutually orthogonal Latin squares is for almost all nonprime values of n. In later classes we'll talk a bit more about what we do know for these cases, and why we might care about these objects (surprisingly: geometry, and/or your internet.)