a tri-colouring modulo $\phi(X)$, κ say. The restriction of κ to $E(H)$ is a tri-colouring of H, since $\phi(X) \cap E(H) = \emptyset$; and so its lift, λ say, via ψ belongs to $\mathscr{C}_1$ and hence to $\mathscr{C}_3$. But for $e \in E(S)$, let $\kappa'(e) = \kappa(\phi(e))$; then κ' is a tri-colouring of S modulo X, and λ is its restriction to R. This contradicts that X is a contract for S, and the result follows. ▌

4. UNAVOIDABILITY

In this section we prove (2.3). A *cartwheel* is a configuration W such that there is a vertex w and two circuits C_1, C_2 of $G(W)$ with the following properties:

(i) $\{w\}$, $V(C_1)$, $V(C_2)$ are pairwise disjoint and have union $V(G(W))$

(ii) C_1 and C_2 are both induced subgraphs of $G(W)$, and $U(C_2)$ bounds the infinite region of $G(W)$

(iii) w is adjacent to all vertices of C_1 and to no vertices of C_2.

It follows that the edges of $G(W)$ are of four kinds: edges of C_1, edges of C_2, edges between w and $V(C_1)$, and edges between $V(C_1)$ and $V(C_2)$. We call w the *hub* of the cartwheel. See Fig. 3.

To avoid confusion, let us stress that Fig. 3 is a picture of W, not of the free completion of W; the free completion would have *three* concentric circuits around w.

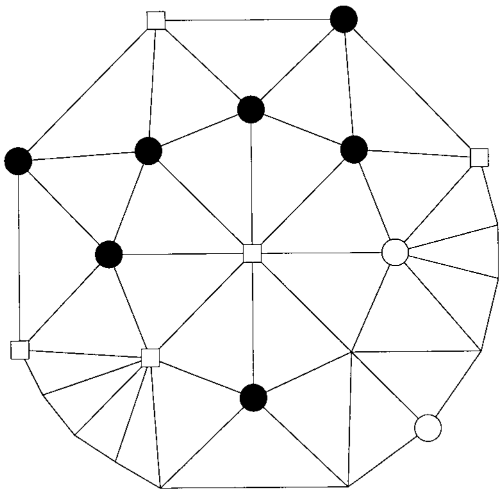

FIG. 3. A cartwheel.