CCS Discrete Math I — Professor: Padraic Bartlett

Homework 2: More Counting

Due Friday, Week 1 — *UCSB 2014*

Do **two** of the **three** problems below! Prove all of your claims. If you've seen some of these problems before, try the ones you have not seen first.

1. Find the flaw in the following proof:

   Theorem 1. *All ponies are the same color.*

   Proof. We proceed by induction. Specifically, let $P(n)$ be the claim "In any collection of n ponies, all of these ponies are the same color."

   Base case: we want to prove $P(1)$. But $P(1)$ is trivially true; in any collection made of one pony, all of the ponies in that set are the same color.

   Inductive case: we want to prove that $P(n) \Rightarrow P(n+1)$. In other words, we want to prove that whenever $P(n)$ is true, $P(n+1)$ is also true. To do this: assume that $P(n)$ is true, i.e. that in any set of n ponies, all of those ponies are the same color. With this assumption, we want to prove that $P(n+1)$ is true: i.e. that in any set of $n+1$ ponies, all of these ponies are also the same color.

   To do this: take any set of $n+1$ ponies, and write them as the set $\{p_1, \ldots p_{n+1}\}$. Break this set up into two subsets of size n: the subset $\{p_1, \ldots p_n\}$ and the subset $\{p_2, \ldots p_{n+1}\}$. These are both sets of size n: by our inductive hypothesis, they are both the same color. But these sets share the ponies $p_2, \ldots p_n$ in common! Therefore, whatever color our first set $\{p_1, \ldots p_n\}$ is **must be** the same color as the second set $\{p_2, \ldots p_{n+1}\}$, because they overlap! Therefore, all of our $n+1$ ponies are the same color, and we've proven that $P(n+1)$ is true (given our assumption $P(n)$.)

   So: we've proven that $P(1)$ is true, and that $P(n) \Rightarrow P(n+1)$. Therefore, by induction, we've proven that our claim $P(n)$ is true for all n; if we let n be the total number of ponies in existence, this proves our claim.

2. Prove that every fourth Fibonacci number is a multiple of 3. (Hint: show that $f_{4k+4} = 5f_{4k} + 3f_{4k-1}$, for any k.)

3. A **lattice path** in the plane $\mathbb{R}^2$ is a path joining integer points via steps of length 1 either upward or rightward. Show that for any $a, b \in \mathbb{N}$, the number of lattice paths from $(0,0)$ to the point (a,b) is $\binom{a+b}{a}$.