CCS Discrete Math I — Professor: Padraic Bartlett

Homework 7: Other Famous Numbers

Due Friday, Week 4 — *UCSB 2014*

Over the past week, we've studied the Catalan numbers, one of the most famous sequences of integers in combinatorics. On this set, you'll study some other famous numbers! Each problem will introduce a class of numbers, and mention one or two interesting properties which you are then invited to prove.

Several problems here have extra-credit parts, or are extra-credit themselves. You do not need to do these parts to get full credit for the problem.

Also, because these problems are all multi-part and this set came out a little late, we're asking you to do just two of these problems, instead of the usual three. Solve **two** of the **five** problems below!

1. Recall, from earlier in class, the definition of a **partition** of a set:

   Definition. Given any set S, a **partition** of S into n pieces is a way to write S as the union of n disjoint sets $A_1, \ldots A_n$, such that all of the A_i sets are mutually disjoint (i.e. none of these sets overlap), none of the A_i are empty, and the union of all of them is our set S.

   For example, if $S = \{1, 2, 3, \text{fish}, 6, 7, 8\}$, a partition of S into three parts could be given by

   $$A_1 = \{1, 8\}, A_2 = \{\text{fish}\}, A_3 = \{2, 3, 6, 7\}.$$

   We define the **Bell number** B_n as the number of different partitions of a set of n elements into any number of parts. For example, $B_2 = 2$ because a set with two elements can be partitioned in exactly two ways:

   $$S = \{1, 2\} : \{1\} \cup \{2\} \text{ or } \{1, 2\}.$$

   (a) Find B_3.

   (b) Show that the Bell numbers satisfy the recurrence relation

   $$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

   (c) Show that B_{n-1} is even if and only if n is a multiple of 3. (That is; $B_2, B_5, B_8, \ldots$ and so on should all be even, and all other Bell numbers should be odd.)

   Hint: Prove the following useful theorem:

   Theorem. For any natural number n, we have $B_{n+2} = B_n + B_{n+1} \mod 2$.

   Hint for this theorem: consider partitions of the set $\{1, 2, \ldots n, a, b\}$. There are three kinds of partitions:

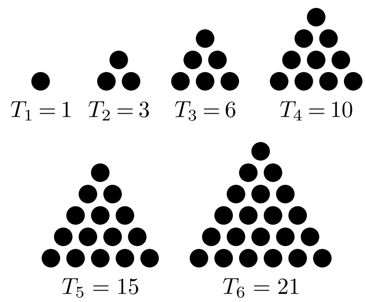
i. Partitions where both a, b land in the same set.

ii. Partitions where a lands in a singleton set $\{a\}$ and b lands in a singleton set $\{b\}$.

iii. Other partitions: i.e. partitions where a, b land in different sets, but at least one of those sets is not a singleton set.

Explain why there are always an even number of these "other" partitions. Conclude from this that $B_{n+2} = B_n + B_{n+1} \mod 2$.

2. The **Triangular numbers** T_n are given by the number of coins that you need to stack in two dimensions to create an equilateral triangle:



(a) Show that $T_n = \frac{n(n+1)}{2}$ for any n.

(b) Take any $n \in \mathbb{N}$. Show that you can write n as a sum of at most six triangular numbers.

Hint, to make this problem actually doable (it was too hard earlier:) you may use the following theorem without proof.

Theorem. (Lagrange.) Any natural number n can be written as the sum of four squares of natural numbers.

3. In class, we studied the Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$.

(a) Prove that if n is a composite[1] number, then f_n is also a composite number, for $n \geq 5$.

(b) Show that the converse does not hold: i.e. find a prime number p such that f_p is not prime.

(c) (Extra credit. Also hard.) Find infinitely many Fibonacci numbers that are primes, or prove that this is impossible.

4. **Pascal's triangle** is a triangular array formed by the binomial coefficients:

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \end{array}$$

[1]A natural number n is composite if we can write $n = lk$, for two other natural numbers k, l.

In general, the n-th row of Pascal's triangle is given by the $n+1$ binomial coefficients

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdots \quad \binom{n}{n-2}\binom{n}{n-1} \quad \binom{n}{n}$$

(a) Prove that in Pascal's triangle, each entry is the sum of the two entries above it.

(b) Let's consider the Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$ again. Prove the identity

$$f_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}.$$

For example,

$$21 = f_8 = \sum_{k=0}^{\lfloor (7-1)/2 \rfloor} \binom{7-k}{k} = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} = 1 + 6 + 10 + 4 = 21.$$

(c) For the Fibonacci numbers f_1, f_2, f_3, f_4, circle for each number the binomial coefficients associated to that Fibonacci number in the sum in (b). What pattern do you see? Explain your observation.

5. The **Pell numbers** are defined by the following recurrence relation:

$$P_0 = 0, P_1 = 1, P_n = 2p_{n-1} + P_{n-2}.$$

The first few Pell numbers are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, \ldots$

(a) Using generating functions, find a closed form for the Pell numbers (like how we did for the Fibonacci sequence.) If you've done this right, there should be $\sqrt{2}$'s in your answer.

(b) An alternate definition for the Pell numbers is the following, based on the idea of "approximating $\sqrt{2}$."
For any k, define $r(k)$ to be an integer such that $\frac{r(k)}{k}$ is as close to $\sqrt{2}$ as possible. For example, $r(1) = 1$, because $1/1$ is the closest fraction to $\sqrt{2}$ that you can make with denominator 1. Similarly, $r(2) = 3$, because $3/2$ is the closest fraction with denominator 2, $r(3) = 4$, $r(4) = 6$, $r(5) = 7$, and so on/so forth.
Look at the sequence given by $\{\frac{r(k)}{k}\}_{k=1}^{\infty}$:

$$\frac{1}{1}, \frac{3}{2}, \frac{4}{3}, \frac{6}{4}, \frac{7}{5}, \ldots$$

Notice that when we do this, some of our approximations are "better" than others. That is: $3/2$ is closer to $\sqrt{2}$ than $\frac{4}{3}$. So: get rid of all of the approximations that are worse than earlier approximations! In other words, form the sequence $\left\{\frac{p_n}{q_n}\right\}_{n=1}^{\infty}$ of "increasingly better approximations," defined as follows: $\frac{p_1}{q_1} = 1$, and

$$\frac{p_n}{q_n} = \text{ the smallest value of } k \text{ such that } \left|\frac{r(k)}{k} - \sqrt{2}\right| < \left|\frac{p_{n-1}}{q_{n-1}} - \sqrt{2}\right|.$$

Look at the first few terms of this sequence:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$$

Funny thing: the denominators of these fractions are the Pell numbers! Explain why this is true.

6. (Extra credit. Harder than the other hard extra credit problems.) Consider the following function $f : \mathbb{N} \to \mathbb{N}$:

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even, or} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Using this function, for any n, we can define its corresponding "hailstone-sequence:"

$$(n, f(n), f(f(n)), f(f(f(n))), \ldots)$$

For example, $n = 6$ has the following corresponding sequence:

$$(6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots)$$

Notice that if we ever get to 1, our sequence "repeats" the $4 - 2 - 1$ loop forever. We call such sequences **terminating,** as we can effectively cut them off when they get to 1 and lose no information.

(a) Show that there are infinitely many terminating numbers.

(b) Either find a nonterminating number, or prove no such number can exist.