CCS Discrete III	Professor: Padraic Bartlett

Homework 4: More Random Walks

Due Friday, Week 3	*UCSB 2015*

Do **three** of the following **five** problems!

1. In class, we proved that the probability of escape on $\mathbb{Z}^3$ was at least 1/6. Improve this bound by proving that $p_{\text{esc}}(\mathbb{Z}^3) \geq 1/3$.

2. Prove $p_{\text{esc}}(\mathbb{Z}^3) \leq 5/6$ (i.e. find an upper bound for our probability of escape!)

3. Take any finite collection $p_1, \ldots p_n$ of polygons. A **tiling** of $\mathbb{R}^2$ with such polygons is any way to place copies of these polygons in the plane, so that they "cover" $\mathbb{R}^2$ and only overlap on their edges. (Formally, you can phrase this property as the request that every point in $\mathbb{R}^2$ is either contained within the edge of at least one polygon, or within the interior of exactly one polygon.)

   Given any such tiling, we can form a graph, where our vertices are the corners of our polygons, and we connect two vertices with a graph-edge if there is a polygonal edge connecting them with no other vertices in between. So, for example, a tiling of the plane with side-length 1 squares corresponds to the $\mathbb{Z}^2$-graph.

   Is there any polygonal tiling of $\mathbb{R}^2$ whose associated graph has a nonzero probability of escape? Or must a random walker on any polygonal tiling graph return to wherever they start with probability 1?

4. Similarly to the above; define a "polytopal tiling" of $\mathbb{R}^3$ with finitely many three-dimensional polytopes $p_1, \ldots p_n$ as a way to cover $\mathbb{R}^3$ with polytopes that only overlap on their edges and faces, and turn these objects into graphs as well. (So, the tiling of $\mathbb{R}^3$ with cubes of side length 1 is precisely the $\mathbb{Z}^3$-graph.)

   Is there any polytopal tiling of $\mathbb{R}^3$ whose associated graph has a **zero** probability of escape? Or, given any polytopal tiling of $\mathbb{R}^3$, does a random walker always have a nonzero chance of escape?

5. Take any graph G with distinguished source vertex s and sink vertex t. Define a **flow** on G as any function $j : V \times V \to \mathbb{R}$ on the edges of G that satisfies Kirchoff's laws at every vertex other than s, t: that is,

   - For any $y \in V(G), y \neq s, t$, $\displaystyle\sum_{y \in N(x)} j_{xy} = 0$.
   - If $\{x, y\} \notin E(G)$, then $j_{xy} = 0$.

   (a) Show that for any circuit C, the current is a flow.

   (b) Prove the following "conservation of energy" property: if $w : V(G) \to \mathbb{R}$ is any function defined on the vertices of our graph and j is any flow on our graph from source s to sink t such that $j_s = \sum_{y \in N(s)} j_{sy} = 1$, then

$$w_s - w_t = \frac{1}{2} \sum_{x,y \in V(G)} (w_x - w_y) \cdot j_{xy}.$$

(c) Take any circuit C. Define the **energy dissipation** of this circuit with respect to any flow $j : V \times V \to \mathbb{R}$ as the following sum:

$$\frac{1}{2} \sum_{x,y \in V(G)} j_{xy}^2 \cdot R_{xy}.$$

Show that the current "minimizes" the total energy dissipation: that is, that if j is any flow on our circuit with $j_s = \sum_{s \in N(y)} j_{sy} = i_s$, then

$$\frac{1}{2} \sum_{x,y \in V(G)} i_{xy}^2 \cdot R_{xy} \leq \frac{1}{2} \sum_{x,y \in V(G)} j_{xy}^2 \cdot R_{xy}.$$

6. Use problem 5 to prove Rayleigh's Monotonicity Theorem:

Theorem 1. *If any of the individual resistances in a circuit increase, then the overall effective resistance of the circuit can only increase or stay constant; conversely, if any of the individual resistances in a circuit decrease, the overall effective resistance of the circuit can only decrease or stay constant.*

Bonus! Find the exact value of $p_{\text{esc}}(\mathbb{Z}^3)$.