CCS Discrete III | Professor: Padraic Bartlett

Homework 5: Flows

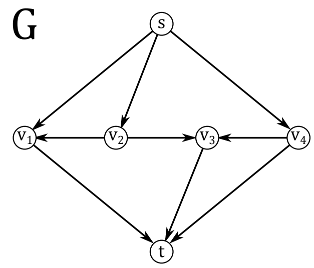
Due Friday, Week 3 | *UCSB 2015*

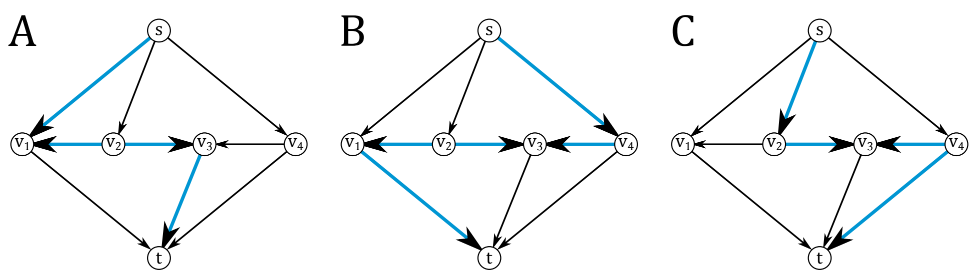
Do **one** of the following **three** problems!

1. In class, we proved that Ford-Fulkerson holds for all rational-valued capacity functions. Extend this result to all real-valued capacity functions, by proving the following theorem:

   Theorem 1. *(Ford-Fulkerson:) Suppose that G is a graph with source and sink nodes s, t and a real-valued capacity function c. Then the maximum value of any feasible flow f on G is equal to the minimum value of any cut on G.*

2. This graph is meant to illustrate the dangers of irrational flows and the Ford-Fulkerson algorithm for finding maximum flows. Consider the following graph G, which we've drawn below along with three distinct paths A, B, C:





   Suppose that the capacity function on this graph has $c(v_2 \to v_1) = 1, c(v_2 \to v_3) = 1, c(v_4 \to v_3) = \Phi = \frac{\sqrt{5}-1}{2}$, and the capacity of all other edges is some really large constant: say the speed of light $(3 \cdot 10^8.)$

   Run Ford-Fulkerson on this graph by picking the following specific augmenting paths:

   (a) At the very start, augment on the path $(s, v_2), (v_2, v_3), (v_3, t)$.

   (b) Now, augment on path B.

   (c) Now, augment on path C.

   (d) Now, augment on path B.

   (e) Now, augment on path A.

(f) Repeat, starting at step (b).

Show that all of these paths are augmenting, and thus that Ford-Fulkerson could conceivably pick these paths; show that the process above converges to a flow with value $2 + \sqrt{5}$, and finally notice that this is far far less than its maximum possible value!

3. Suppose that you have a $n \times n$ matrix of real numbers A. Let c_i denote the sum of all of the elements in the i-th column of A, and r_i denote the sum of all of the entries in the i-th row of A. A **rounding** of A is the act of taking each value a_{ij}, r_i, c_j and rounding these numbers either up or down to integer values. A rounding is called **successful** if in the resulting rounded matrix A_R, the row and column sums are the same things as the values we chose to round the r_i, c_j's to. We give an example of a successful and an unsuccessful rounding below:

$$\begin{array}{|c|c|c|c|}\hline 0.6 & 0.8 & 2.7 & 4.1 \\ \hline 0.3 & 1.9 & 2.7 & 4.9 \\ \hline 2.3 & 0.4 & 0.4 & 3.1 \\ \hline 3.2 & 3.1 & 5.8 & \\ \hline \end{array} \xrightarrow[\text{rounding}]{\text{unsuccessful}} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 5 \\ \hline 0 & 2 & 3 & 5 \\ \hline 2 & 0 & 0 & 3 \\ \hline 3 & 3 & 6 & \\ \hline \end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 0.6 & 0.8 & 2.7 & 4.1 \\ \hline 0.3 & 1.9 & 2.7 & 4.9 \\ \hline 2.3 & 0.4 & 0.4 & 3.1 \\ \hline 3.2 & 3.1 & 5.8 & \\ \hline \end{array} \xrightarrow[\text{rounding}]{\text{successful}} \begin{array}{|c|c|c|c|}\hline 1 & 0 & 3 & 4 \\ \hline 1 & 2 & 2 & 5 \\ \hline 2 & 1 & 0 & 3 \\ \hline 4 & 3 & 5 & \\ \hline \end{array}.$$

Prove, ideally using the Max-Flow-Min-Cut theorem, that every real-valued $n \times n$ matrix has a successful rounding.