CCS Discrete II	Professor: Padraic Bartlett

Homework 12: Ramsey Theory

Due Friday, Week 8 — *UCSB 2015*

Do **three** of the six problems listed here! Have fun, prove all of your claims, and let me know if you have any questions!

1. Take the graph K_7. Color each of the edges of K_7 either green or blue. We know, by Ramsey's theorem, that there is at least one triangle in this graph with edges all the same color, because $R(3,3) = 6 < 7$. Prove that there are actually at least **four** triangles in this graph such that each triangle's edges are all the same color.

2. In class, we mentioned that finding $R(3,3)$ can be thought of as a game:

   - There are two players, Red and Blue. Their gameboard consists of six points drawn on a plane. Players alternate turns, and Red starts first.
   - On a given player's turn, they must connect two points that do not have a line drawn between them yet, with a line of their given color.
   - The game ends when a monochromatic triangle is drawn on the board, in which case that player loses.

   We proved in class that this game always ends with one player losing (i.e. no draws are possible), because $R(3,3) = 6$. Assuming that both players play optimally, which player always wins: red or blue? (Warning: hard!)

3. Suppose you play the game above on a gameboard with just **five** points. Prove that with optimal play from both players, this game will always end in a draw.

4. We defined the **tri-color Ramsey number** $R(k,k,k)$ to be the smallest natural number n such that in any (red, green, blue)-coloring of K_n, there is a monochromatic K_k of some color. For any $k \geq 3$, prove that

$$R(k,k,k) \geq \lfloor 3^{k/2} \rfloor.$$

5. (a) Show that for any graph G on at least $5 \cdot n$ vertices, either G or $\overline{G}$ must contain n disjoint triangles, for any $n \geq 3$.

   (b) For any n, find a graph G on $5n - 1$ vertices such that neither G or $\overline{G}$ contain n disjoint triangles, for any $n \geq 3$.

6. (a) Draw any five points in the plane, so that no three line on a single line. Prove that there is a subset consisting of four of these points, such that they are the vertices of a convex quadrilateral.

   (b) Prove the following theorem:

Theorem. For any m, there is some value N_m such that the following holds: suppose that we draw any set of N_m points in the plane, so that no three line on a single line. Then there is a subset consisting of m of these points, such that they are the vertices of a convex m-gon.