CCS Discrete II	Professor: Padraic Bartlett

Homework 16: The Unit Distance Graph

Due Friday, Week 10 — *UCSB 2015*

Do **one** of the four problems listed here! Have fun, prove all of your claims, and let me know if you have any questions!

1. Finish our proof that $\chi(\mathbb{Q}^2)$ is rational by proving the following claims:
   (a) Take any two rational numbers $\frac{a}{b}, \frac{c}{d}$ written so that $GCD(a, b) = 1 = GCD(c, d)$. Suppose that $a^2d^2 + b^2c^2 = b^2d^2$.

   Prove that both b and d must be odd.

   (b) Using (a), take any two points $\vec{q}, \vec{r} \in \mathbb{Q}^2$. Show that if $d(\vec{q}, \vec{r}) = 1$, then $\vec{q} \sim \vec{r}$ under the equivalence relation defined in class on Monday.

   (c) Take the collection of all rational points equivalent to $(0, 0)$, under the equivalence relation defined in class. Color points in this class red when they are of the form $\left(\frac{odd}{odd}, \frac{odd}{odd}\right)$ or $\left(\frac{even}{odd}, \frac{even}{odd}\right)$, and color them blue otherwise; i.e. color points blue when they are of the form $\left(\frac{odd}{odd}, \frac{even}{odd}\right)$ or $\left(\frac{even}{odd}, \frac{odd}{odd}\right)$.

   Prove the claim we made in class: any two points that are distance 1 apart in this class are colored different colors.

2. Zorn's lemma, in mathematics, says the following:

   Lemma 1 (Zorn's lemma)**.** *Suppose that $(P, \leq)$ is a nonemptu partially ordered set*[1] *with the following property: in any totally ordered subset*[2] *T of P, there is an upper bound u: i.e. an element u such that for any other $a \in T$, $a \leq t$.*

   Then P has a maximal element: i.e. there is some $m \in P$ such that we never have $m < a$ for any $a \in P$.

   Prove this, using the axiom of choice.

3. Let $\mathbb{Q}^3$ denote the graph on $\mathbb{Q}^3$ formed by connecting $\vec{v}, \vec{w}$ with an edge if and only if $d(\vec{v}, \vec{w}) = 1$. Show that $\chi(\mathbb{Q}^3) = 2$.

---

[1] A partially ordered set $(P, \leq)$ is a set P and a binary relation $\leq$ such that $\leq$ satisfies the folowing three axioms:

- $a \leq a$ (reflexivity)
- $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry)
- $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

[2] A subset T of P is called totally ordered if every two elements are comparable in T: i.e. for any $a, b \in T$, $a \leq b$ or $b \leq a$.