CCS Discrete II	Professor: Padraic Bartlett

Homework 17: Various Odds and Ends (Extra Credit)

Due Friday, Finals Week *UCSB 2015*

This set is extra-credit! If you want to receive said credit for this set, finish it and get it to my office (or my email if you're leaving before then) by 11:30am on Friday.

Also this set is fun! (Recall from previous sets the relative difficulty level of "fun".) Do as many as you want; credit will be given for up to three correct answers.

1. Prove that the chromatic number of space is at least 5. (By "space," we mean $\mathbb{R}^3$, where we connect two points with an edge if and only if they are distance one apart.

2. (Do not do this problem if you haven't done problem 1 yet!) Read the paper "On the space chromatic number," by Oren Nechustan:

   http://www.sciencedirect.com/science/article/pii/S0012365X00004064

   Summarize its proof (i.e. give a writeup that you would use for a lecture to your peers) that $\chi(\mathbb{R}^3) \geq 6$.

3. Prove that the chromatic number of space is at most 21. (By "space," we mean $\mathbb{R}^3$, where we connect two points with an edge if and only if they are distance one apart.

4. Using Zorn's lemma, prove the De Bruijn-Erdös Compactness theorem:

   Theorem. Suppose that k is a natural number and G is a graph such that every finite subgraph of G can be properly vertex-colored with at most k colors.

   Then G itself can be properly vertex-colored with at most k colors.

5. Take any coloring of the plane $\mathbb{R}^2$, pick any of our colors, and let C be the collection of all points in the plane that are colored that color. We say that C **realizes** distance s if there are two points in C that are distance s apart, and say that it **forbids** distance s otherwise.

   We say that a coloring of the plane $\mathbb{R}^2$ is **polychromatic** if and only if no single color realizes every possible distance. We define the **polychromatic number of the plane** as the smallest value k such that there is a k-coloring of the plane that is polychromatic, and denote this value as $\chi_p(\mathbb{R}^2)$. Prove that $\chi_p(\mathbb{R}^2) \leq \chi 6$.

6. Prove that $\chi_p(\mathbb{R}^2) \geq 3$. (The best known is slightly stronger, that $\chi_p(\mathbb{R}^2) \geq 4$. I don't know if 3 makes the problem easier yet, but the proof for 4 is certainly doable if difficult.)