CCS Discrete II	Professor: Padraic Bartlett

Homework 4: Matrices

Due Friday, Week 3	*UCSB 2015*

This problem set is structured somewhat differently. Do the **three** problems in the required section! (I switched up the style here because there are a few things I really want you to do here in order for matrices to make sense.) As always, prove all of your claims, and have fun!

1 Required Problems

Take any field $\mathbb{F}$. In class, we proved that $\mathbb{F}^n$ was a vector space over $\mathbb{F}$; common examples of this vector space were things like $\mathbb{R}^n$ over $\mathbb{R}$, or $(\mathbb{Z}/2\mathbb{Z})^n$ over $\mathbb{Z}/2\mathbb{Z}$.

Given this setup, we define a **matrix** as follows:

Definition. A $n \times n$ **matrix** A over a field $\mathbb{F}$ is a particular kind of function from $\mathbb{F}^n \to \mathbb{F}^n$. A matrix is specified by giving a $n \times n$ array of elements from $\mathbb{F}$, drawn as follows:

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

With this array of elements specified, we can define the function $A : \mathbb{F}^n \to \mathbb{F}^n$ as follows: for any vector $\vec{v} = (v_1, v_2, \dots v_n) \in \mathbb{F}^n$, write

$$A(\vec{v}) = \left(\sum_{i=1}^n a_{1i}v_i, \ \sum_{i=1}^n a_{2i}v_i, \ \sum_{i=1}^n a_3 v_i, \ \dots\ , \sum_{i=1}^n a_{ni}v_i \right).$$

For example, the following object

$$A = \begin{bmatrix} \pi & \pi & 0 \\ 0 & 2 & 1 \\ 1 & 3 & -21 \end{bmatrix}$$

can be thought of as a 3×3 matrix over $\mathbb{R}$, and thus is a function from $\mathbb{R}^3 \to \mathbb{R}^3$. If we wanted to know where A sends the vector $(1, 2, 1)$, we could just use the definition above to calculate

$$\begin{aligned} A((1,2,1)) &= \left(\sum_{i=1}^3 a_{1i}v_i, \ \sum_{i=1}^3 a_{2i}v_i, \ \sum_{i=1}^3 a_3 v_i \right) \\ &= (\pi \cdot 1 + \pi \cdot 2 + 0 \cdot 1, 0 \cdot 1 + 2 \cdot 2 + 1 \cdot 1, 1 \cdot 1 + 3 \cdot 2 + (-21) \cdot 1) \\ &= (3\pi, 5, -14). \end{aligned}$$

1. (Warm-ups.) To get used to the above notation, calculate the following values:

   (a) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, thought of as a matrix over $\mathbb{R}$, find $A((0,0,0))$, $A((3,4,2))$, and $A((1,-2,1))$.

   (b) For $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, thought of as a matrix over $\mathbb{Z}/2\mathbb{Z}$, find $A((0,0,0))$ and $A((1,0,1))$.

   (c) Take any field $\mathbb{F}$; because $\mathbb{F}$ is a field, it contains an additive identity 0 and a multiplicative identity 1. Look at the $n \times n$ matrix A defined by putting 1 on the diagonal and 0's elsewhere: in other words,

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

   Show that for any $\vec{v} = (v_1, v_2, \dots v_n) \in \mathbb{F}^n$, we have $A((\vec{v})) = \vec{v}$.

   (d) Again, take any field $\mathbb{F}$. Consider the "all-zeroes" $n \times n$ matrix A, given by

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

   Show that for any $\vec{v} = (v_1, v_2, \dots v_n) \in \mathbb{F}^n$, we have $A((\vec{v})) = \vec{0}$.

2. Take any two vectors $\vec{v}, \vec{w} \in \mathbb{F}^n$. Define the **dot product** of $\vec{v} \cdot \vec{w}$ as follows:

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i.$$

   For example, the dot product of $(1,2,3)$ and $(2,2,4)$, if both are thought of as vectors in $(\mathbb{Z}/5\mathbb{Z})^3$, is

$$1 \cdot 2 + 2 \cdot 2 + 3 \cdot 4 = 18.$$

   Take any $n \times n$ matrix A over a field $\mathbb{F}$. Let $\vec{a_{r_i}}$ denote the i-th row of A: that is, $\vec{a_{r_i}} = (a_{i,1}, a_{i,2}, \dots a_{i,n})$.

   Prove that for any vector $\vec{v}$, we have

$$A(\vec{v}) = \left(\vec{a_{r_1}} \cdot \vec{v}, \vec{a_{r_2}} \cdot \vec{v}, \dots \vec{a_{r_n}} \cdot \vec{v}\right).$$

3. Take any two $n \times n$ matrices A, B over a field $\mathbb{F}$. We think of these two objects as maps $\mathbb{F}^n \to \mathbb{F}^n$; consequently, we can talk about **composing** these maps: that is, we can form the map $B \circ A : \mathbb{R}^n \to \mathbb{R}^n$ that sends a vector $\vec{v}$ to $B(A(\vec{v}))$.

For example, if $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ are two matrices over $\mathbb{R}$, then we can find $(B \circ A)((-1, 1, 0))$ as follows:

$$\begin{aligned} (B \circ A)((-1,1,0)) &= B(A((-1,1,0))) \\ &= B((1 \cdot -1 + 2 \cdot 1 + 1 \cdot 0, 2 \cdot -1 + 1 \cdot 1 + 2 \cdot 0, 1 \cdot -1 + 1 \cdot 1 + 0 \cdot 0)) \\ &= B((1, -1, 0)) \\ &= (-1 \cdot 1 + -2 \cdot -1 + -3 \cdot 0, 0 \cdot 1 + 0 \cdot -1 + 0 \cdot 0, 0 \cdot 1 + 1 \cdot -1 + 3 \cdot 0) \\ &= (1, 0, -1). \end{aligned}$$

(a) For $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$ a pair of 2×2 matrices over $\mathbb{R}$, find $(B \circ A)((1, 1))$ and $(B \circ A)((-2, 0))$.

(b) Find two matrices A, B and a vector $\vec{v}$ such that $(B \circ A)(\vec{v}) \neq (A \circ B)(\vec{v})$. (In other words, the order of composition is important when working with matrices!)

(c) Take any two $n \times n$ matrices A, B over some field F:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,n} \\ b_{2,1} & b_{2,2} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,n} \end{bmatrix}.$$

Denote the row vectors of B as $\vec{b_{r_i}}$'s and the column vectors of A as $\vec{a_{c_j}}$'s.

Consider the following $n \times n$ matrix C, made by placing $\vec{b_{r_i}} \cdot \vec{a_{c_j}}$ in each entry (i, j):

$$C = \begin{bmatrix} \vec{b_{r_1}} \cdot \vec{a_{c_1}} & \vec{b_{r_1}} \cdot \vec{a_{c_2}} & \dots & \vec{b_{r_1}} \cdot \vec{a_{c_n}} \\ \vec{b_{r_2}} \cdot \vec{a_{c_1}} & \vec{b_{r_2}} \cdot \vec{a_{c_2}} & \dots & \vec{b_{r_2}} \cdot \vec{a_{c_n}} \\ \dots & \dots & \ddots & \dots \\ \vec{b_{r_n}} \cdot \vec{a_{c_1}} & \vec{b_{r_n}} \cdot \vec{a_{c_2}} & \dots & \vec{b_{r_n}} \cdot \vec{a_{c_n}} \end{bmatrix}.$$

Take any vector $\vec{v} \in \mathbb{F}^n$. Prove that $(B \circ A)(\vec{v}) = C(\vec{v})$. (This process is what we refer to by "matrix multiplication;" we will usually denote the matrix C above as $B \cdot A$.)