CCS Discrete II	Professor: Padraic Bartlett

Homework 5: Graphs

Due Friday, Week 4	*UCSB 2015*

Pick **three** of **five** problems to solve. Prove all of your claims, and have fun!

1. Define the **unit distance graph**, denoted in the literature as $\mathbb{R}^2$, as follows:

   - Vertex set: the collection of all points in $\mathbb{R}^2$.
   - Edge set: connect two vertices if their corresponding points in $\mathbb{R}^2$ are distance 1 apart in the plane.

   This problem asks you to bound the chromatic number of $\mathbb{R}^2$.

   (a) One easy lower bound is that you'll need at least three colors. This is because there is an equilateral triangle with side length 1 in $\mathbb{R}^2$, and therefore if we're coloring all of $\mathbb{R}^2$ we'll need to give those three points different colors, or we'll have an edge with monochromatic endpoints.
   Improve this bound by 1: i.e. prove that $\chi(\mathbb{R}^2) \geq 4$.

   (b) Find any finite upper bound on the chromatic number of $\mathbb{R}^2$: i.e. find some $n \in \mathbb{N}$ such that $\chi(\mathbb{R}^2) \leq n$.

2. Prove that all "map-graphs" are planar.

3. A sequence $d_1 \geq d_2 \geq \ldots d_n$ of nonnegative integers is called **graphic** if and only if there is a graph G on n vertices such that $\deg(v_i) = d_i$, for every $v_i \in V(G)$.

   Determine whether any of the following sequences are graphic:

   - $5, 3, 3, 2, 2, 2$.
   - $6, 2, 2, 2$.
   - $3, 2, 2, 2, 1, 1, 1$
   - $3, 3, 3, 3, 3, 3, 3, 3, 3, 3$
   - $\underbrace{n, n, n \ldots n}_{n+1 \text{ times}}$.

4. A graph G is called **Eulerian** if it contains a path P that satisfies the following two properties:

   - P starts and ends on the same vertex.
   - P uses every edge in G exactly once.

   Show that a graph G is Eulerian if and only if the degree of every vertex in G is even.

5. For a graph G, let $\Delta(G)$ denote the maximum degree of all of the vertices in G. Prove that if G is a graph, then $\chi(G) \leq \Delta(G) + 1$.