CCS Discrete II	Professor: Padraic Bartlett
Lecture 10: The Unit Distance Graph	
Week 10	*UCSB 2015*

1 The Unit Distance Graph Problem

One of the first graphs we studied in this class was the **unit distance graph**:

Definition. Consider the following method for turning $\mathbb{R}^2$ into a graph:

- Vertices: all points in $\mathbb{R}^2$.
- Edges: connect any two points (a, b) and (c, d) iff the distance between them is exactly 1.

This graph is called the **unit distance graph,** which we denote as $\mathbb{R}^2$.

On the homework, we proved the following result:

Theorem. $4 \leq \chi(\mathbb{R}^2) \leq 7$.

We review one possible proof of this claim here.

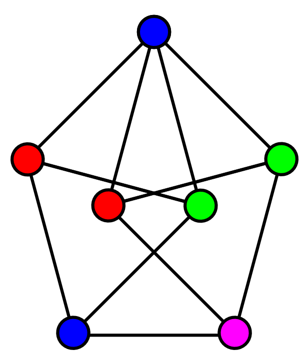
Proof. To get a lower bound, it suffices to consider finite graphs G that we can draw in the plane using only straight edges of length 1. Because our graph on $\mathbb{R}^2$ must contain any such graph "inside" of itself, we know that $\chi(G) \leq \chi(\mathbb{R}^2)$ (as any coloring of $\mathbb{R}^2$ with k colors will always induce a coloring of G with k colors as well.)

By examining a equilateral triangle T, which has $\chi(T) = 3$, we can see that

$$\chi(\mathbb{R}^2) \geq 3.$$

This is because it takes three colors to color an equilateral triangle's vertices in such a way that no edge has two endpoints of the same color.

Similarly, by examining the following pentagonal construction (called a Moser spindle,)



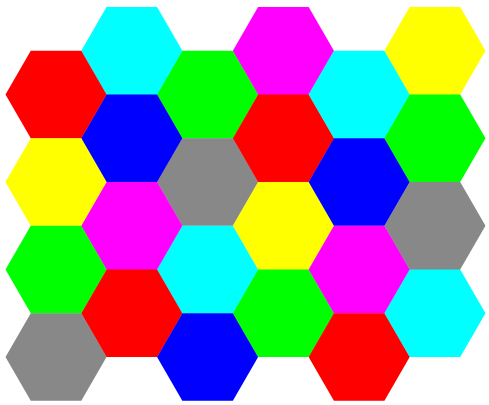
we can actually do one better and say that

$$\chi(\mathbb{R}^2) \geq 4.$$

Verify for yourself that you can't color this graph with three colors (it's a simple case-checking exercise.)

Conversely: to exhibit an upper bound on $\chi(\mathbb{R}^2)$ of k, it suffices to create a way to color the plane with k colors in such a way that no two points distance 1 apart get the same color. Because the chromatic number is simply the smallest value of k for which a coloring exists, showing that we have any proper k-coloring proves that we have $\chi(\mathbb{R}^2) \leq k$.

So: consider the following way to color the plane!



To be specific: start by tiling the plane with hexagons of diameter slightly less than 1. Then, color the hexagons with seven colors as described above; i.e. repeat the color pattern

gray, red, teal, yellow, blue, green, magenta

on each strip of hexagons, shifted two colors over for each strip. This gives you a mesh of hexagons, so that any two hexagons of the same color are at least more than distance 1 apart. Therefore, any line segment of length 1 cannot bridge two different hexagons of the same color! As well, because the hexagons have diameter slightly less than one, no line segment of length 1 can lie entirely within a hexagon of the same color. Therefore, there are no line segments of length 1 with both endpoints of the same color.

In other words, we have just proven that this is a proper coloring of the plane! So we can color the plane with seven colors: i.e. we just showed that

$$\chi(\mathbb{R}^2) \leq 7.$$

□

These bounds on $\chi(\mathbb{R}^2)$ were not too crazy to find: it took us about a page to get them, with a decent amount of those pages dedicated to being careful with our wording and notation. As a result, we might hope that finding $\chi(\mathbb{R}^2)$'s actual value is something we could easily finish with a few more pages of work.

This. . . is not the case. Not even a little bit. In fact, finding the chromatic number of the plane — formally called the Hadwiger-Nelson problem, if you're reading through textbooks or articles — has withstood attacks from the best minds in combinatorics since the 1950's, and to this day no better bounds than the ones we've established are known.

So, it's not too likely that we're going to be able to solve[1] this problem in this class. If we were going to try, though, how would we attempt to come up with a solution? Typically, when presented with an open or difficult problem, mathematicians rarely attempt to **directly** solve the problem; if this was likely to succeed, someone probably would have done it already! Instead, what we do is try to create a related problem to the one we want to study; we either take a special case of the original problem, or remove some conditions from it, or attempt to get a weaker conclusion, or other such things.

For the unit distance graph problem, a natural question to ask is the following: if we can't find the chromatic number of $\mathbb{R}^2$, maybe we can find the chromatic number of other spaces! For example, consider the following vertex sets, which we turn into graphs by connecting all vertices at distance 1:

1. $\mathbb{R}^1$: i..e the real line! This object has chromatic number 2; simply color the points in intervals of the form $[2k, 2k+1)$ red, and points in intervals of the form $[2k+1, 2k+2)$ blue. Any two points at distance 1 cannot lie in the same interval, nor can they span two nonadjacent intervals, so this is a proper 2-coloring!

2. $\mathbb{Z}^k$, for any k: i.e. the integer lattice! This also has chromatic number 2: simply color every point of the form $(x_1, \ldots x_k)$ with $\sum x_i =$ even red, and every point with $\sum x_i =$ odd blue. Any two points that are distance 1 apart are identical in all but one of their coordinates, at which they differ by 1; therefore those two points must be different colors, and thus we have a 2-coloring.

3. $\mathbb{R}^3$: i.e. three-dimensional space! This turns out to be harder than the unit distance problem. Using arguments similar to the ones we used for two-dimensional space, we can see that $6 \leq \chi(\mathbb{R}^3) \leq 15$; try to check this out on your own!

4. $\mathbb{Q}^2$: i.e. the rational plane! On one hand, this seems like it should be similar to $\mathbb{R}^2$: they're both dense two-dimensional spaces full of points that seem tricky to color! On the other hand, unlike $\mathbb{R}^2$, we cannot construct things like equilateral triangles; so it is not obvious how to even get a lower bound of 3. To work on this, we need the following concept, which should be familiar to most of you:

2 Equivalence Relations

In this section, we review the definitions and several key examples of equivalence relations. If you remember them, feel free to skip forward to the next section!

Definition. Take any set S. A **relation** R on this set S is a map that takes in ordered pairs of elements of S, and outputs either true or false for each ordered pair.

You know many examples of relations:

[1] Try it, though! Someone will eventually solve this problem. Why not you?

- Equality ($=$), on any set you want, is a relation; it says that $x = x$ is true for any x, and that $x = y$ is false whenever x and y are not the same objects from our set.

- "Mod n" ($\equiv \mod n$) is a relation on the integers: we say that $x \equiv y \mod n$ is true whenever $x - y$ is a multiple of n, and say that it is false otherwise.

- "Less than" ($<$) is a relation on many sets, for example the real numbers; we say that $x < y$ is true whenever x is a smaller number than y (i.e. when $y - x$ is positive,) and say that it is false otherwise.

- "Beats" is a relation on the three symbols (rock, paper, scissors) in the game Rock-Paper-Scissors. It says that the three statements "Rock beats scissors," "Scissors beats paper," and "Paper beats rock" are all true, and that all of the other pairings of these symbols are false.

In this class, we will study a specific class of particularly nice relations, called **equivalence relations**:

Definition. A relation R on a set S is called an **equivalence relation** if it satisfies the following three properties:

- **Reflexivity**: for any $x \in S$, xRx.

- **Symmetry**: for any $x, y \in S$, if xRy, then yRx.

- **Transitivity**: for any $x, y, z \in S$, if xRy and yRz, then xRz.

It is not hard to classify our example relations above into which are and are not equivalence relations:

- Equality ($=$) is an equivalence relations on any set you define it on – it trivially satisfies our three properties of reflexivity, symmetry and transitivity.

- "Mod n" ($\equiv \mod n$) is an equivalence relation on the integers. This is not hard to check:
  - **Reflexivity**: for any $x \in \mathbb{Z}$, $x - x = 0$ is a multiple of n; therefore $x \equiv x \mod n$.
  - **Symmetry**: for any $x, y \in S$, if $x \equiv y \mod n$, then $x - y$ is a multiple of n; consequently $y - x$ is also a multiple of n, and thus $y \equiv x \mod n$.
  - **Transitivity**: for any $x, y, z \in S$, if $x \equiv y \mod n$ and $y \equiv z \mod n$, then $x - y$, $y - z$ are all multiples of n; therefore $(x - y) + (y - z) = x - y + y - z = x - z$ is also a multiple of n, and thus $x \equiv z \mod n$.

- "Less than" ($<$) is not an equivalence relation on the real numbers, as it breaks reflexivity: $x \not< x$, for any $x \in \mathbb{R}$.

- "Beats" is not an equivalence relation on the three symbols (rock, paper, scissors) in the game Rock-Paper-Scissors, as it breaks symmetry: "Paper beats rock" is true, while "Rock beats paper" is false.

Equivalence relations are remarkably useful because they allow us to work with the concept of equivalence classes:

Definition. Take any set S with an equivalence relation R. For any element $x \in S$, we can define the **equivalence class** corresponding to x as the set

$$\{s \in S \mid sRx\}$$

Again, you have worked with lots of equivalence classes before. For $\mod 3$ arithmetic on the integers, for example, there are three possible equivalence classes for an integer to belong to:

$$\{\ldots -6, -3, 0, 3, 6 \ldots\}$$
$$\{\ldots -5, -2, 1, 4, 7 \ldots\}$$
$$\{\ldots -4, -1, 2, 5, 8 \ldots\}$$

Every element corresponds to one of these three classes.

The concept of equivalence classes is useful largely because of the following observation:

Observation. Take any set S with an equivalence relation R. On one hand, every element x is in some equivalence class generated by taking all of the elements equivalent to x, which is nonempty by reflexivity. On the other hand, any two equivalence classes must either be completely disjoint or equal, by symmetry and transitivity: if the sets $\{s \in S \mid sRx\}$ and $\{s' \in S \mid s'Ry\}$ have one element t in common, then tRx and tRy implies, by symmetry and transitivity, that xRy; therefore, by transitivity, any element in one of these equivalence relations must be in the other as well.

Consequently, these equivalence classes **partition** the set S: i.e. if we take the collection of all distinct equivalence classes, every element of S is in exactly one such set.

One particularly useful use of the concept of equivalence classes is in our definition of the rational numbers themselves! In particular, ask yourself: what is the set of the rational numbers?

Most people will quickly say something equivalent to the following:

$$\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

The issue with this as a set is that it has **lots** of different entries for numbers that we usually think are not different objects! I.e. the set above contains

$$\frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots,$$

all of which we think are the same number! People usually then go back and change our definition above to the following:

$$\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b > 0, GCD(a, b) = 1 \right\}.$$

This fixes our issue from earlier: we no longer have "duplicated" numbers running around. However, it has other issues: suppose that you wanted to define addition on this set! Naively, you might hope that the following definition would work:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

However, for many fractions, the output of this operation is not an element of our new set!

$$\frac{2}{5} + \frac{8}{5} = \frac{40+10}{25} = \frac{50}{25} \notin \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b > 0, GCD(a,b) = 1 \right\}.$$

These difficulties that we're running into with the rational numbers come from the fact that, practically speaking, **they aren't a set** in most contexts that we work with them! Rather, they are a **set with an equivalence relation**:

- The underlying set for the rational numbers: $\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}$.
- The equivalence relation: we say that $\frac{a}{b} = \frac{c}{d}$ if there are a pair of integers k, l such that $ka = lc$ and $kb = ld$.
- A **rational number** is any equivalence class of our set above under the above equivalence relation. This is the idea we have when we think of

  $$\frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots$$

  as all representing the "same number" 1/2: we're identifying 1/2 with its equivalence class!
- In this setting, we define addition, multiplication, and all of our other properties just how we would normally: i.e. we define

  $$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd},$$

  where the only wrinkle is that by each of $\frac{a}{b}, \frac{c}{d}, \frac{ad+bc}{bd}$ we actually mean "take any element equivalent to these fractions," and by equality above we actually mean our equivalence relation.

With this detour completed, we return to our original problem:

3 The Chromatic Number of $\mathbb{Q}^2$

Theorem. The chromatic number of $\mathbb{Q}^2$ is 2.

Proof. Our proof proceed in a few steps, using equivalence relations. Some of the details here are left as exercises on the HW: check them out!

First, consider the following relation $\sim$ on $\mathbb{Q}^2$: for any two points $(a_1, a_2), (b_1, b_2) \in \mathbb{Q}^2$, we say that $(a_1, a_2) \sim (b_1, b_2)$ if and only the following happens: $a_1 - b_1$ and $a_2 - b_2$ both have odd denominators. (Because rational numbers, as discussed above, have many possible representatives, we avoid ambiguity by asking that we write these fractions with the GCD of their numerator and denominator equal to 1, and where the denominator is positive.)

For example, $\left(\frac{1}{2}, \frac{1}{3}\right) \not\sim \left(\frac{3}{4}, \frac{5}{6}\right)$, because their difference $\left(\frac{1}{2} - \frac{3}{4}, \frac{1}{3} - \frac{5}{6}\right) = \left(\frac{-1}{4}, \frac{1}{2}\right)$ does not consist of fractions whose denominators are odd. However, $\left(\frac{7}{15}, \frac{4}{3}\right) \sim \left(\frac{2}{15}, \frac{1}{9}\right)$, because their difference $\left(\frac{7}{15} - \frac{2}{15}, \frac{4}{3} - \frac{1}{9}\right) = \left(\frac{1}{3}, \frac{11}{9}\right)$ consists of fractions whose denominators are odd.

It is not hard to see that this is an equivalence relation:

- **Reflexivity**: for any $\left(\frac{a}{b}, \frac{c}{d}\right) \in \mathbb{Q}^2$, $\left(\frac{a}{b}, \frac{c}{d}\right) - \left(\frac{a}{b}, \frac{c}{d}\right) = (0, 0)$. 0, when written as a fraction with the GCD of its numerator and denominator equal to 1 and with a positive denominator, has a unique representation as $\frac{0}{1}$. 1, in particular, is odd; so the difference of any point in $\mathbb{Q}^2$ with itself consists of a pair of fractions with odd denominators! Therefore any point in $\mathbb{Q}^2$ is related to itself under $\sim$: i.e. $\sim$ is reflexive.

- **Symmetry**: for any $\left(\frac{a}{b}, \frac{c}{d}\right), \left(\frac{e}{f}, \frac{g}{h}\right) \in \mathbb{Q}^2$, we want $\left(\frac{a}{b}, \frac{c}{d}\right) \sim \left(\frac{e}{f}, \frac{g}{h}\right)$ if and only if $\left(\frac{e}{f}, \frac{g}{h}\right) \sim \left(\frac{a}{b}, \frac{c}{d}\right)$. But this is trivial; the difference of these two pairs of fractions are equal up to the sign! Therefore, the denominators of the differences of one pair are odd if and only if the denominators of the differences of the other pair are odd; so our relation $\sim$ is symmetric.

- **Transitivity**: take any $\left(\frac{a}{b}, \frac{c}{d}\right), \left(\frac{e}{f}, \frac{g}{h}\right), \left(\frac{i}{j}, \frac{k}{l}\right) \in \mathbb{Q}^2$, and suppose that $\left(\frac{a}{b}, \frac{c}{d}\right) \sim \left(\frac{e}{f}, \frac{g}{h}\right)$ and $\left(\frac{e}{f}, \frac{g}{h}\right) \sim \left(\frac{i}{j}, \frac{k}{l}\right)$. Then we have

$$\left(\frac{a}{b}, \frac{c}{d}\right) - \left(\frac{e}{f}, \frac{g}{h}\right) = \left(\frac{\boxed{?}_1}{odd_1}, \frac{\boxed{?}_2}{odd_2}\right), \text{ and}$$

$$\left(\frac{e}{f}, \frac{g}{h}\right) - \left(\frac{i}{j}, \frac{k}{l}\right) = \left(\frac{\boxed{?}_3}{odd_3}, \frac{\boxed{?}_4}{odd_4}\right).$$

Adding these two equations together gives us

$$\left(\frac{a}{b}, \frac{c}{d}\right) - \left(\frac{e}{f}, \frac{g}{h}\right) + \left(\frac{e}{f}, \frac{g}{h}\right) - \left(\frac{i}{j}, \frac{k}{l}\right) = \left(\frac{\boxed{?}_1}{odd_1}, \frac{\boxed{?}_2}{odd_2}\right) + \left(\frac{\boxed{?}_3}{odd_3}, \frac{\boxed{?}_4}{odd_4}\right).$$

$$\Rightarrow \qquad \left(\frac{a}{b}, \frac{c}{d}\right) - \left(\frac{i}{j}, \frac{k}{l}\right) = \left(\frac{\boxed{?}_1 \cdot \boxed{?}_2}{odd_1 \cdot odd_3}, \frac{\boxed{?}_3 \cdot \boxed{?}_4}{odd_2 \cdot odd_4}\right).$$

In other words: we have that the difference of $\left(\frac{a}{b}, \frac{c}{d}\right), \left(\frac{i}{j}, \frac{k}{l}\right)$ consists of a pair of fractions with odd denominators! In particular, this means that when we write either of these fractions in lowest common terms, the denominators will **stay** odd, as dividing out common factors from the top and bottom cannot **introduce** a factor of 2 to the bottom.

This gives us $\left(\frac{a}{b}, \frac{c}{d}\right) \sim \left(\frac{i}{j}, \frac{k}{l}\right)$, and therefore that our relation is transitive.

The reason we care about this equivalence relation is the following observation:

Observation. If two points in $\mathbb{Q}^2$ are distance one apart, then they are equivalent under the equivalence relation $\sim$.

Proof. On the HW! $\square$

This observation is very useful for our goals. In specific, suppose that we can do the following:

- For any equivalence class E of $\mathbb{Q}^2$ under $\sim$, suppose that we can create a proper 2-coloring of E's vertices.

Equivalence classes are all disjoint; therefore, if we have a coloring of each individual equivalence class, we can union them all together to get a coloring of the whole space! Moreover, we know that two points in $\mathbb{Q}^2$ can be connected by an edge only if they are equivalent under $\sim$; consequently, if each equivalence class is colored properly, then we have no chance of getting a "conflict" when combining them all together (and we thus have a proper 2-coloring of the whole graph $\mathbb{Q}^2$!)

To prove our claim, then, it suffices to show how to properly 2-color the vertices of any equivalence class E. We start this by first noticing that all of our equivalence classes are actually the "same" up to translation:

Observation. Let E denote the equivalence class containing everything equivalent to $(0,0)$ under the relation $\sim$. Take any other point $(p_1, q_1) \in \mathbb{Q}^2$. Then the set

$$E_{(p_1,q_1)} = \{(e_1, e_2) + (p_1, q_1) \mid (e_1, e_2) \in E\}$$

is actually the equivalence class containing everything equivalent to (p_1, q_1) under the relation $\sim$. In other words, every equivalence class under our relation is just a copy of E, translated by some constant.

Proof. This is not very difficult. First, take any two points $(e_1, e_2)+(p_1, q_1), (e_3, e_4)+(p_1, q_1)$ in $E_{(p_1,q_1)}$. Notice that because (e_1, e_2) and (e_3, e_4) are in the same equivalence class (and are thus equivalent to each other!), we have

$$(e_1, e_2) + (p_1, q_1) - ((e_3, e_4) + (p_1, q_1)) = (e_1, e_2) - (e_3, e_4) = \left(\frac{\boxed{?}_1}{odd_1}, \frac{\boxed{?}_2}{odd_2}\right).$$

This demonstrates that any two points in $E_{(p_1,q_1)}$ are equivalent. Now, take any point (s_1, t_1) equivalent to (p_1, q_1). Notice that because

$$\begin{aligned}(0,0) - ((s_1, t_1) - (p_1, q_1)) &= ((p_1, q_1) - (p_1, q_1)) - ((s_1, t_1) - (p_1, q_1))\\ &= (p_1, q_1) - (s_1, t_1)\\ &= \left(\frac{\boxed{?}_1}{odd_1}, \frac{\boxed{?}_2}{odd_2}\right),\end{aligned}$$

we can express (s_1, t_1) as an element that is equivalent to $(0,0)$ plus (p_1, q_1). This demonstrates that every element that is equivalent to (p_1, q_1) lies in the set $E_{(p_1,q_1)}$.

By combining these two results, we have show that $E_{(p_1,q_1)}$ is the equivalence class corresponding to (p_1, q_1), as desired. $\square$

Why is this nice? Well, it means that if we can color just **one** equivalence class — say, E, the equivalence class corresponding to $(0,0)$ — then we can color **every** equivalence class by just taking our coloring of E and translating it by a constant! As stated before, we could then take all of these colorings and combine them together to get a coloring of all of $\mathbb{Q}^2$ without conflicts.

Therefore, it suffices to simply describe how to color E. We do this with one final observation:

Observation. Take the set E consisting of all of the points in $\mathbb{Q}^2$ equivalent to $(0,0)$ under the equivalence relation $\sim$. Color all points of the form $\left(\frac{odd_1}{odd_2}, \frac{odd_3}{odd_4}\right)$ and $\left(\frac{even_1}{odd_2}, \frac{even_3}{odd_4}\right)$ red, and all points of the form $\left(\frac{odd_1}{odd_2}, \frac{even_3}{odd_4}\right)$ and $\left(\frac{even_1}{odd_2}, \frac{odd_3}{odd_4}\right)$ blue.

Then no two points in E that are distance 1 apart are colored the same color.

Proof. On the HW! □

This concludes our proof: we have created a 2-coloring of the rational plane! Which isn't quite the same as coloring the real plane. However, many of the techniques that we've developed here will be useful to us in later talks and attempts to solve this problem! Progress. □

We close our class with one of the most beautiful and strange results in mathematics I know:

4 The Chromatic Number of the Plane and the Axiom of Choice

As with many strange results in mathematics, the axiom[2] of choice is somehow involved. In case you haven't seen this before, here is the axiom of choice:

Axiom of Choice For every family Φ of nonempty sets, there is a choice function

$$f : \Phi \to \bigcup_{S \in \Phi} S,$$

such that $f(S) \in S$ for every $S \in \Phi$.

Roughly speaking, this axiom says that for any collection of sets, we can pick one element out of each set.

When this was first proposed as an axiom, mathematicians were opposed to it on several grounds:

- Constructivist and intutionist mathematicians opposed it, on the grounds that it posits the existence of functions without any clue whatsoever as to how to find them!

[2] An axiom, in case you haven't seen this word before, is something that mathematicians simply assume is true about the world. For example, the Axiom of Union says that if we have two sets A, B, we can form the set $A \cup B$. Axioms like this, and moreover proofs that only use axioms that are this simple, are attempts to write down the simplest, most basic assumptions about what mathematics is; this is mostly so that when horrible, terrifying counterintuitive results creep up, we can at least know there wasn't any way around it.

- Many other working mathematicians just thought it was a true statement; i.e. that AC was a trivial consequence of any logical framework of mathematics.

Surprisingly enough, however, Paul Cohen and Kurt Gödel proved that the axiom of choice is independent of the Zermelo-Fraenkel axioms of set theory, the current framework within which we do mathematics: i.e. that it is its own proper axiom! Pretty much all of modern mathematics accepts the Axiom of Choice; it's a pretty phenomenally useful axiom, and most fields of mathematics like to be able to call on it when pursuing nonconstructive proofs.

There are, however, a number of disconcerting "paradoxes" that arise from working within ZFC, the framework of axioms given by the Zermelo-Fraenkel axioms + the axiom of choice:

- The well-ordering principle: the statement that any set S admits a well-ordering[3] One consequence of this is that there's a way to order the real numbers so that they "locally" look like the natural numbers! This is strange.
- The Banach-Tarski paradox: there's a way to chop up and rearrange a sphere into two spheres of the same surface area. This is also strange.
- The existence of nonmeasurable sets: There are bounded subsets of the real line to which we cannot assign any notion of "length," given that we want length to be a translation-invariant, nontrivial, and additive function on subsets of $\mathbb{R}$. This is also also strange.

Motivated by these strange results, Solovay (a set theorist) introduced the following two "alternate" axioms to choice:

- ($AC_{\aleph_0}$, the countable axiom of choice): For every **countable** family Φ of nonempty sets, there is a choice function
$$f : \Phi \to \bigcup_{S \in \Phi} S,$$
such that $f(S) \in S$ for every $S \in \Phi$.
- (LM, Lebesgue-measurability): Every bounded set in $\mathbb{R}$ is measurable[4].

From here, he proved that there are models of mathematics in which we can use and study these axioms in place of AC:

[3]A well-ordering on a set S is a relation $\leq$ such that the following properties hold:

- (antireflexive:) $a \leq b$ and $b \leq a$ implies that $a = b$.
- (total:) $a \leq b$ or $b \leq a$, for any $a, b \in S$.
- (transitive:) $a \leq b, b \leq c$ implies that $a \leq c$.
- (least-element:) Every nonempty subset of S has a least element.

[4]Measure is a precise notion of the idea of 'length" extended to sets other than intervals. We want any good notion of measure to be translation-invariant (i.e. shifting an interval should not change its length), nontrivial (i.e. the length of $[0, 1]$ should be 1), and additive (i.e. the length of the union of two disjoint objects should be the sum of their lengths.) See our notes from last quarter's Introduction to Higher Mathematics for more on this!

Theorem 1 (Solovay's Theorem)**.** *There are models of mathematics in which ZF + LM + $AC_{\aleph_0}$ all hold.*

For brevity's sake, we will denote ZF + the axiom of choice by ZFC, and ZF + LM + $\text{AC}_{\aleph_0}$ by ZFS.

5 $\chi(\mathbb{R}^2)$ in ZFS

Fun fact $\chi(\mathbb{R}^2)$ may depend on AC. The "may" in that statement comes from the fact that we currently don't know what $\chi(\mathbb{R}^2)$ **is**, and therefore it's difficult to say what it depends on. However, consider the following graph G:

- $V(G) = \mathbb{R}$,
- $E(G) = \{(s,t) : s - t - \sqrt{2} \in \mathbb{Q}\}$.

This graph has a chromatic number using the ZFC axiom system: call it $\chi^{ZFC}(G)$. It has some other chromatic number under the ZFS axiom system: call it $\chi^{ZFS}(G)$. The following two theorems show that these two chromatic numbers are remarkably distinct:

Theorem 2. $\chi^{ZFC}(G) = 2$.

Proof. Let

$$S = \{q + n\sqrt{2} | q \in \mathbb{Q}, n \in \mathbb{Z}\}.$$

Define an equivalence relation $\sim$ on $\mathbb{R}$ as follows:

$$x \sim y \text{ if and only if } x - y \in S.$$

Under the relation $\sim$, the elements of $\mathbb{R}$ are broken up into various sets of elements that are all "equivalent" to each other under the relation $\sim$. Let $\{E_i\}$ be the collection of all of these equivalence classes.

Using the axiom of choice, pick one element y_i from each set E_i, and collect all of these elements in a single set E. Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \text{ the unique element } y_i \text{ in } E \text{ such that } x \sim y_i.$$

Now, define a two-coloring of $\mathbb{R}$ as follows: for any $x \in \mathbb{R}$,

- color x red if and only if there is an odd integer n such that

$$x - f(x) - n\sqrt{2} \in \mathbb{Q}, \text{ and}$$

- color x blue if and only if there is an even integer n such that

$$x - f(x) - n\sqrt{2} \in \mathbb{Q}.$$

Why does this assign a unique color to every vertex? Well, we know that $x \sim f(x)$; so $x - f(x)$ is always of the form $q + n\sqrt{2}$, by construction. n is always either even or odd: so the above two cases assigns every vertex a unique color.

Furthermore, we claim that this coloring is a **proper** coloring: i.e. that no edge has two endpoints of the same color. To see why, take any edge $\{x, y\}$. By definition, we have to have $x - y = q + \sqrt{2}$, for some q. This means that $x \sim y$: i.e. that $f(x) = f(y)$! Consequently, we can rewrite

$$x - y = q + \sqrt{2}$$

as

$$\Rightarrow (x - f(x)) + (y - f(y)) = q + \sqrt{2}.$$

So: we know that there is some pair of integers m, n such that $x - f(x) - n\sqrt{2}$ and $y - f(y) - \sqrt{2}$ are both in $\mathbb{Q}$. By looking at the above equation, we can furthermore see that because there is exactly one copy of $\sqrt{2}$ on the right hand side above, these integers must have different parities! But this is just the statement that x and y have different colors under our coloring.

Therefore, this is a proper 2-coloring, and the chromatic number of G under ZFC is 2. □

Without the axiom of choice, however, it turns out that we need a few more colors to color G:

Theorem 3. *For G as above, $\chi^{ZFS}(G) > |\mathbb{N}| = \aleph_0$.*

Proof. Consider the following massive hammer from analysis:

Theorem 4. *(Lebesgue Density Theorem) If a set A has nonzero measure, then there is an interval I such that*

$$\frac{\mu(A \cap I)}{\mu(I)} \geq 1 - \epsilon,$$

for any $\epsilon > 0$.

A proof of this theorem is very far beyond the scope of this class. A rough idea, however, is the following: given any set, we can approximate its measure by coming up with a collection of intervals that cover the set, and using this as an upper bound. One theorem you can prove is that if a set has measure d, you can come up with some collection of intervals that cover that set with measure only slightly more than d. Therefore, if you take one of those intervals, you'd expect the ratio of the "measure" of the set elements in that interval to the length of that interval to be very close to 1.

We mention it because it helps us prove the following lemma:

Lemma 5. *If $A \subset [0,1]$ and A doesn't contain a pair of adjacent vertices in G, then A has measure*[5] *0.*

Proof. So: choose any set A of measure > 0, and pick I such that

$$\frac{\mu(A \cap I)}{\mu(I)} \geq 99/100,$$

for instance. Then, pick $q \in \mathbb{Q}$ such that $\sqrt{2} < q < \sqrt{2} + \mu(I)/100$, and define $B = \{x - q + \sqrt{2} : x \in A\}$. Then B has been translated by at most 1/100-th of the length of I: so we have that

$$\frac{\mu(B \cap I)}{\mu(I)} \geq 98/100.$$

So, because $(A \cap I) \cup (B \cap I) \subset I$, and both of these sets are almost all of I, we know that they must overlap! In other words, there's an element y in both A and B. However, if $y \in B$, we know that there is some $x \in A, q \in \mathbb{Q}$ such that $y = x - q + \sqrt{2}$.

This statement is precisely the claim that there's an edge between x and y! Therefore, A contains an edge, as claimed. $\square$

With this lemma, our proof is pretty straightforward. Suppose that we could color $\mathbb{R}$ with $\aleph_0$-many colors, and that the collection of colors used is given by the collection $\{A_i\}_{i=1}^{\infty}$. Let $B_i = A_i \cap [0,1]$; then we have that all of the B_i are disjoint and $\bigcup B_i = [0,1]$. Consequently, we have that $\sum_{i=1}^{\infty} \mu(B_i) = \mu([0,1]) = 1$; so at least one of the B_i's have to have nonzero measure! This contradicts our above lemma; consequently, no such $\aleph_0$-coloring can exist. $\square$

Math! What. I don't even.

[5]The measure of a set S is defined as the infimum of the sum $\sum_{i=1}^{\infty}(a_i, b_i)$, where we range over all countable collections of intervals $\{(a_i, b_i)\}$ such that $\bigcup(a_i, b_i) \supset S$. We denote this number by writing $\mu(S)$.