CCS Problem-Solving I — Professor: Padraic Bartlett

# Homework 10: Infinite Descent

*Due Thursday, Week 5, at the start of class.* — *UCSB 2014*

Solve **one** of the following **three** problems. As always, prove your claims!

These problems are all centered around the concept of **infinite descent**. Specifically, notice that the natural numbers have the following property:

**Property.** There is no infinite decreasing sequence of natural numbers $\{a_n\}_{n=0}^{\infty}$.

This is not hard to see: if we start at some value $a_0 \in \mathbb{N}$ and we're always decreasing, we have at most $a_0$-more terms in our sequence before we get to 0, at which point we can no longer decrease!

This is a rather simple observation that is used in a number of clever problems; we list three here!

0. Solve any un-signed-up-for problems from HW#9!

1. Prove that there are no three integers $a, b, c > 0$ such that

   $$a^2 - b^2 = 2abc.$$

   (Hint: look at $ab$ and its factors!)

2. A sequence $\{a_n\}_{n=0}^{\infty}$ is called an $k$-**arithmetic progression** if the difference between consecutive terms $a_{n+1} - a_n$ is equal to $k$. For example,

   $$1, 4, 7, 10, 13, 16, \ldots$$

   is an 3-arithmetic progression.

   Prove that for $k \neq 0$, there is no $k$-arithmetic progression $\{a_n\}_{n=0}^{\infty}$ made entirely out of perfect squares

3. Find all of the prime numbers $p$ such that the following holds: there are positive integers $a, b$ and $n$ such that

   $$p^n = x^3 + y^3.$$

   (Hint: factor $x^3 + y^3$, and use the fact that $p$ is a prime.)