CCS Problem-Solving I	Professor: Padraic Bartlett

Homework 11: Various Number Theory Tricks

Due Thursday, Week 6, at the start of class. *UCSB 2014*

Solve **three** of the following **six** problems. As always, prove your claims!

Also: due to Veteran's Day, there is no class this Tuesday! Therefore, this set is due Thursday. Enjoy the holiday!

Themes for this problem set:

- Divisibility and primes! It is sometimes useful to consider factorizations of numbers. (You might have noticed this already.)
- Mod n! It's also useful.
- Fermat's little theorem! For any n, $n^p \equiv n \mod p$.
- Wilson's theorem: for any p, $(p-1)! \equiv -1 \mod p$.

You can use those last two results without proof on the following problems!

0. Solve any un-signed-up-for problems from HW#10!

1. Prove or disprove the following claim: there is no positive integer value of n such that the set

$$\{n, n+1, n+2, n+3, n+4, n+5\}$$

can be broken into two sets, with the product of the elements of the first set equal to the product of the elements in the second set.

2. (a) Take any prime number p and any positive integer n. Look at the quantity $n!$. Then, the number of times that p occurs as a factor in $n!$ is given by the sum

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots$$

For example, the number of times that 2 occurs as a factor of 5! is 3, as $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8 \cdot 15$; and indeed,

$$\begin{aligned} &\left\lfloor \frac{5}{2} \right\rfloor + \left\lfloor \frac{5}{4} \right\rfloor + \left\lfloor \frac{5}{8} \right\rfloor + \dots \\ =&2 + 1 + 0 + 0 + \dots \\ =&3. \end{aligned}$$

Prove this fact!

(b) Find all of the values of n such that $n!$ ends in 100 zeroes.

3. Suppose that n is a perfect square such that its last four digits are all equal. Prove that they are all equal to 0.
4. Show that for every prime p, there is an integer n such that $2^n + 3^n + 6^n - 1$ is a multiple of p.
5. Suppose that there is some positive integer n such that the **first** digit of $2^n, 5^n$ are the same. What is this digit? Prove your claim.
6. How many primes p exist with the following property: if we write p in base 10, then its digits are alternating 1's and 0's? (For example, 101 is such a prime!)