CCS Problem-Solving I	Professor: Padraic Bartlett

Homework 12: Sequences

Due Tuesday, Week 7, at the start of class. *UCSB 2014*

Themes for this problem set: the concepts of sequences and limits! Here are some ideas you will find useful in these problems. You may use all of these results without proof. You may not use any **other** results without proof, however! In particular, **don't use L'Hopital's theorem.**

- A sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to some value λ if the a_n's "go to λ" at infinity. More formally: $\lim_{n\to\infty} a_n = \lambda$ if and only if for any distance ϵ, there is some cutoff point N such that for any n greater than this cutoff point, a_n is within ϵ of our limit λ.

  In symbols:
  $$\lim_{n\to\infty} a_n = \lambda \text{ iff } (\forall\epsilon)(\exists N)(\forall n > N)\ |a_n - \lambda| < \epsilon.$$

- **Additivity of sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then
  $$\lim_{n\to\infty} a_n + b_n = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n).$$

- **Multiplicativity of sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then
  $$\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n).$$

- **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value λ, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \le c_n \le b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to λ. This is particularly useful for sequences with things like sin(horrible things) in them, as it allows you to "ignore" bounded bits that aren't changing where the sequence goes.

- **Monotone and bounded sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded below and nonincreasing, it also converges. If a sequence is monotone, this is usually the easiest way to prove that your sequence converges, as both monotone and bounded are "easy" properties to work with. One interesting facet of this property is that it can tell you that a sequence converges without necessarily telling you what it converges to!

- **Cauchy sequences**: We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m > n \ge N$, we have
  $$|a_m - a_n| < \epsilon.$$

  You can think of this condition as saying that Cauchy sequences "settle down" in the limit – i.e. that if you look at points far along enough on a Cauchy sequence, they all get fairly close to each other.

  The Cauchy theorem, in this situation, is the following: a sequence is Cauchy if and only if it converges.

Solve **three** of the following **six** problems. As always, prove your claims!

0. Solve any un-signed-up-for problems from HW#10!

1. Can you find sequences of positive integers $\{a_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty}\left(\sqrt{a_k}-\sqrt{b_k}\right)=\pi?$$

2. Let $a_0=\sqrt{2}$, and define a_n recursively by the relation

$$a_n=\sqrt{2+a_{n-1}}.$$

Either find the limit

$$\lim_{k\to\infty} a_n,$$

or show it does not exist.

3. Let $p(x)=x^2-3x+2$. Prove or disprove the following claim: for any $n\in\mathbb{N}$, there are unique integer values a_n, b_n such that the polynomial $q_n(x)=x^n-a_nx-b_n$ is a multiple of $p(x)$.

4. Take a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers such that

$$\lim_{n\to\infty} 2a_{n+1}-a_n=L.$$

Show that $\lim_{n\to\infty} a_n=L$.

(Note: while it is tempting to simply rewrite this limit of a sum as a sum of limits, you can only do this if you can prove that the individual summed limits exist! In other words: while $\lim_{n\to\infty}(n-n)$ exists and is 0, the difference $(\lim_{n\to\infty} n)-(\lim_{n\to\infty} n)=\infty-\infty$ is undefined! Be careful here.)

5. Find the limit

$$\lim_{n\to\infty}\sqrt[n]{n}.$$

6. For any two distinct positive integers x, y, determine the following limits:

$$\text{(i)}\qquad \lim_{n\to\infty}\frac{x^n-y^n}{x^n+y^n}.$$

$$\text{(ii)}\qquad \lim_{n\to\infty}\left(x^n+y^n\right)^{1/n}.$$