CCS Problem-Solving I	Professor: Padraic Bartlett

Homework 15: Probability

Due Tuesday, Week 9, at the start of class. *UCSB 2014*

Solve **one** of the following **three** problems. As always, prove your claims/have fun! Also, this set is **extra-credit**, because of the holiday! Take care, and I'll see you all in a week! –Paddy

This set is themed around the concept of **probability**. Suppose that we have some event, like

A: "This coin, when flipped, will land face-up."

B: "This toast, when flipped, will land butter-side-down."

C: "This cat, when flipped, will always land on its feet."

A thing we often want to determine, as mathematicians, is the **probability** that any event has of occurring. For an event X, we denote this as $P(X)$. For example, in event A, suppose that we are flipping a completely fair coin: that is, a coin that comes up heads exactly half of the time, and comes up tails exactly half of the time. We would write $P(A) = 1/2$ in such a situation!

Given some collection of events, we often want to determine whether they all happen at the same time. For instance, suppose that we have three completely fair coins and want to know the probability that when we flip all three of them, we get three heads. Call the event that coin i comes up heads A_i; then, intuitively, we would assume that our probabilities "multiply!" That is, we would assume that

$$\begin{aligned} P(\text{flip three fair coins, get three heads}) &= P(A_1 \wedge A_2 \wedge A_3) \\ &= P(A_1) \cdot P(A_2) \cdot P(A_3) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}. \end{aligned}$$

However: note that this multiplicative property does not always hold! For example, consider events B, C above. It is well-known that $P(B) = P(C) = 1$. However, consider the "buttered-cat paradox," in which we tape our piece of toast butter-side-up onto the back of a cat. When we flip this toast/cat hybrid, we are measuring $P(B \wedge C)$; and yet, we have

$$P(B \wedge C) = 0 \neq P(B) \cdot P(C).$$

This is a silly example, but it illustrates a key phenomena; not all events obey this multiplicative law! We say that two events D, E are **independent** if they obey this law: that is, D, E are called independent if and only if

$$P(D \wedge E) = P(D) \cdot P(E).$$

Try some of the following problems on probability!

0. Solve any un-signed-up-for problems from HW#14!
1. Find three events A, B, C that collectively satisfy the following properties:
   - Any two events here are **pairwise** independent: that is, $P(A \wedge B) = P(A)P(B)$, $P(A \wedge C) = P(A)P(C)$, and $P(B \wedge C) = P(B)P(C)$.
   - These events are not all simultaneously independent: that is, $P(A \wedge B \wedge C) \neq P(A)P(B)P(C)$.
2. Find the probability that in a group of n people, there are two with the same birthday. Find the smallest value of n for which this is greater than 50%. Find the probability that there are two people in this class with the same birthday!
3. Suppose that a bakery has made $3,141$ pumpkin pies and 271 poisonous pies in preparation for Thanksgiving. On the day before Thanksgiving, the bakery has a ton of undiscerning customers come through. Each customer does the following:
   - The customer picks out two pies at random from all of the bakery's pies, without paying attention to what pies they have.
   - If the customer has chosen two of the same pie, they're kinda dumb and just buy both pies.
   - However, if the customer has one pumpkin pie and one poisonous pie, they look at the two pies, realize that they'd probably rather just have the pumpkin pie, and put the poisonous pie back / buy just the one pumpkin pie.

   This process continues until we are left with either one or no pies (at which point people cannot buy pies two at a time, and they sadly leave.) What are the odds we end up with no pies? How about one pumpkin pie? How about one poisonous pie?