CCS Problem-Solving I — Professor: Padraic Bartlett

Homework 2: Induction/Recursion

Due Thursday, Week 1, at the start of class. — *UCSB 2014*

Solve **one** of the following **three** problems. As always, prove your claims!

0. Do any of the problems that no-one solved on the previous set!

1. Let S be a set of n distinct real numbers. Suppose that A_S is the collection of all of the values given by averaging two distinct elements in S; that is,

$$A_S = \left\{ \frac{x+y}{2} \;\middle|\; x, y \in S, x \neq y \right\}.$$

   For example, if $S = \{1, 3, 5, 7\}$, we have $A_S = \{2, 3, 4, 5, 6\}$; alternately, if $S = \{1, 2, 3, 5\}$, we have $A_S = \{1.5, 2, 2.5, 3, 3.5, 4\}$.

   For any given $n \in \mathbb{N}$, what is the smallest possible size for the set A_S?

2. Take any $n \in \mathbb{N}$ that is greater than zero. A **pleasant decomposition**[1] of n is any way in which we can write n as the sum $a_1 + \ldots + a_k$ of positive integers, for arbitrary k, such that

   - $a_1 \leq a_2 \leq a_3 \leq \ldots a_k$, and
   - $a_k \leq a_1 + 1$.

   For example, $n = 5$ has five different pleasant decompositions:

$$\begin{aligned} 5 &= 5, \\ &= 2+3, \\ &= 1+2+2, \\ &= 1+1+1+2, \text{ and} \\ &= 1+1+1+1+1. \end{aligned}$$

   For any given $n \in \mathbb{N}$, how many pleasant decompositions does n have?

3. Take any positive rational number p/q. Show that we can write p/q as the quotient of products of factorials of prime numbers (where these primes may be repeated.) For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3!}$$

[1] I made this term up!