CCS Problem-Solving I — Professor: Padraic Bartlett

Homework 6: Polynomials

Due Thursday, Week 3, at the start of class. — *UCSB 2014*

Solve **one** of the following **three** problems. As always, prove your claims!

The theme for these problems: polynomials! Sometimes, a problem will boil down to being able to manipulate algebraic expressions until they look like something useful. These three problems involve this task!

0. Solve any un-signed-up-for problems from HW#5!

1. Take any integer n. Show that we can write n as the sum of 5 perfect cubes.

   (For example: $1 = 1^3 - 1^3 + 1^3 - 1^3 + 1^3$; $2 = 0^3 + 2^3 + (-2)^3 + 1^3 + 1^3$. There can be many different ways to write any n as such a sum.)

2. Suppose that $f(x)$ is a polynomial with integer coefficients; moreover, suppose that $f(a) = f(b) = f(c) = -1$, for three distinct integers a, b, c. Prove that there is no integer d such that $f(d) = 0$.

3. (An old USAMO problem.) Consider the equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0.$$

   Because this is a degree-4 polynomial, it has four roots a, b, c, d (where some of these roots may be repeated; i.e. $x^2 - 2x + 1 = 0$ has the root $x = 1$ repeated twice.) Suppose that you know that $ab = -32$. Find k.