CCS Problem-Solving I — Professor: Padraic Bartlett

Homework 7: Inequalities (Triangle, AMGM, $x^2 \geq 0$)

Due Tuesday, Week 4, at the start of class. — *UCSB 2014*

Solve **three** of the following **six** problems. As always, prove your claims!

The theme for these problems: algebraic inequalities! There are many useful algebraic inequalities that come up all the time when solving problems. We list three commonly-occuring ones here:

- For any real number x, $x^2 \geq 0$. This can be generalized slightly: if x, y are two numbers with the same sign, then $xy \geq 0$.
- (Triangle inequality[1].) Take any three points $\vec{x} = (x_1, \ldots x_n), \vec{y} = (y_1, \ldots y_n), \vec{z} = (z_1, \ldots z_n) \in \mathbb{R}^n$. Then the distance[2] $d(\vec{x}, \vec{y})$ from $\vec{x}$ to $\vec{y}$ is bound above by the sum $d(\vec{x}, \vec{z}) + d(\vec{z}, \vec{y})$.
- (Arithmetic mean-geometric mean inequality, i.e. AM-GM.) The arithmetic mean of n nonnegative real numbers is bounded below by their geometric mean. In other words, for any nonnegative numbers $x_1, \ldots x_n \in \mathbb{R}$, we have the following inequality:

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$

Try using them to solve some of the problems below, and have fun!

0. Solve any un-signed-up-for problems from HW#6!
1. Take any three positive real numbers x, y, z. Suppose that for any positive integer n, there is a triangle with side lengths x^n, y^n, z^n. Prove that at least two of the values in $\{x, y, z\}$ are equal.
2. Prove the AM-GM inequality. (If you do this problem, do not use Wikipedia/texts, as they will likely contain the solution.)
3. Suppose that x, y, z are the three side lengths of a triangle with perimeter 2. Show that
$$1 < xy + yz + zx - xyz \leq \frac{28}{27}.$$
4. Let $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ denote two lists of nonnegative real numbers. Show that
$$(a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{1/n} + (b_1 \cdot b_2 \cdot \ldots \cdot b_n)^{1/n} \leq ((a_1 + b_1) \cdot (a_2 + b_2) \cdot \ldots \cdot (a_n + b_n))^{1/n}.$$

[1] We call this the **triangle** inequality because geometrically, it is the following statement: the sum of the lengths of two sides of a triangle is never shorter than the length of the triangle's third side.

[2] If you haven't seen this before: the distance from $(x_1, \ldots x_n)$ to $(y_1, \ldots y_n)$ is the quantity $\sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$. Think about why this makes sense!

5. Suppose that a, b, c are three real numbers such that $|a-b| \geq |c|, |b-c| \geq |a|$ and $|c-a| \geq |b|$. Prove that one of these three numbers is equal to the sum of the other two.

6. Find[3]

$$\min_{a,b\in\mathbb{R}} \left(\max\left((a^2+b, b^2+a)\right)\right).$$

   Prove that your claimed minimum is correct.

[3] If you haven't seen this before: the notation

$$\min_{x\in A} f(x)$$

just means "the minimum value that $f(x)$ can attain on the set A. For example, the minimum value that $(x-1)^2$ can attain on $\mathbb{N}$ is 0; therefore, we would write $\min_{x\in\mathbb{N}} x^2 = 0$. The expression max denotes maximum in the same way.