CCS Problem-Solving I	Professor: Padraic Bartlett

Homework 8: Operations on Sets

Due Thursday, Week 4, at the start of class. *UCSB 2014*

Solve **one** of the following **three** problems. As always, prove your claims!

The theme for these problems: **binary operations** on sets! Specifically: a **binary operation** on a set S is any map that takes elements of $S \times S$ to S; addition, multiplication, exponentiation, the binomial coefficient and other objects are all examples of such operations on $\mathbb{N}$, for example. Often, we will want to study properties of such maps.

For example, suppose that that I gave you a binary operation $\cdot$ on a set S with the following two properties:

- **Identity**: There is a unique element $e \in S$ such that $e \cdot x = x \cdot e = x$, for all $x \in S$.
- **Inverses**: For any $x \in S$, there is a unique $x^{-1} \in S$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$.
- **Annihilator**:[1] There is a unique element $n \in S$ such that $n \cdot x = x \cdot n = n$, for all $n \in S$.

What could you tell me about S? Well: look at our annihilating element n. By inverses, we know that there is an element n^{-1} such that $n^{-1}n = nn^{-1} = e$. But, by definition, we also know that $n^{-1}n = n$, because n is an annihilating element. Therefore, we have $n = e$. But we know that for any $x \in S$, we have $e \cdot x = x$, and that for any $x \in S$, we have $n = n \cdot x$; consequently, because $n = e$, we have $n = x$ for any x. In other words, our group S consists of only one element, n!

This is the kind of thing these problems study: given an operation with some simple properties, what can you conclude?

0. Solve any un-signed-up-for problems from HW#6!

1. Let $\star$ be a binary operation on the set S that satisfies the following three properties:

   - **Commutative**: $\forall a, b \in S$, $a \star b = b \star a$.
   - **Associative**: $\forall a, b, c \in S$, $a \star (b \star c) = (a \star b) \star c$.
   - **Delicious**[2]: $\forall a, b \in S, \exists z \in S$ such that $a \star z = b$.

   Prove that $\star$ satisfies the following **cancellation** property:

   - **Cancellation**: if $a, b, c \in S$ and $a \star c = b \star c$, then $a = b$.

2. Take any set S and any binary operation $\clubsuit$ on S. Suppose that $\clubsuit$ satisfies the following property:

   - **Tartan**[3]: For all $a, b \in S$, we have $(a \clubsuit b) \clubsuit a = b$.

[1] I did not make this term up.
[2] I did make this term up!
[3] I also made this term up!

Prove that ♣ also satisfies the following property:

- **Plaid**[4]: For all $a, b \in S$, we have $a♣(b♣a) = b$.

3. Take any set S and any binary operation $*$ on S that satisfies the following properties:

   - **Associative**: $\forall a, b, c \in S,\ a * (b * c) = (a * b) * c$.
   - **Dubious**: For any $a, b \in S$, we have $a * b = b * a \Rightarrow a = b$.

   Show that it satisfies the following property:

   - **Poisonous**: For any $a, b, c \in S$, we have $a * (b * c) = a * c$.

   Can you find an example of such an operation on $\mathbb{N}$?

[4] Really, if it looks ridiculous, I probably made it up.