Crossing Game-pt 2

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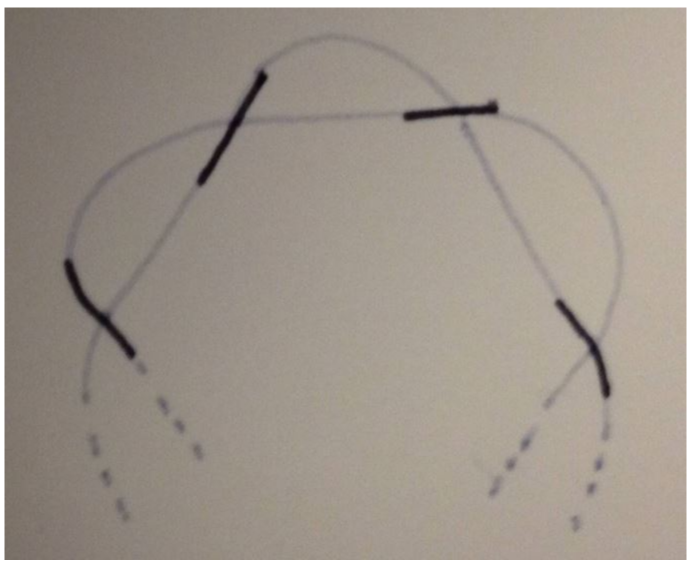
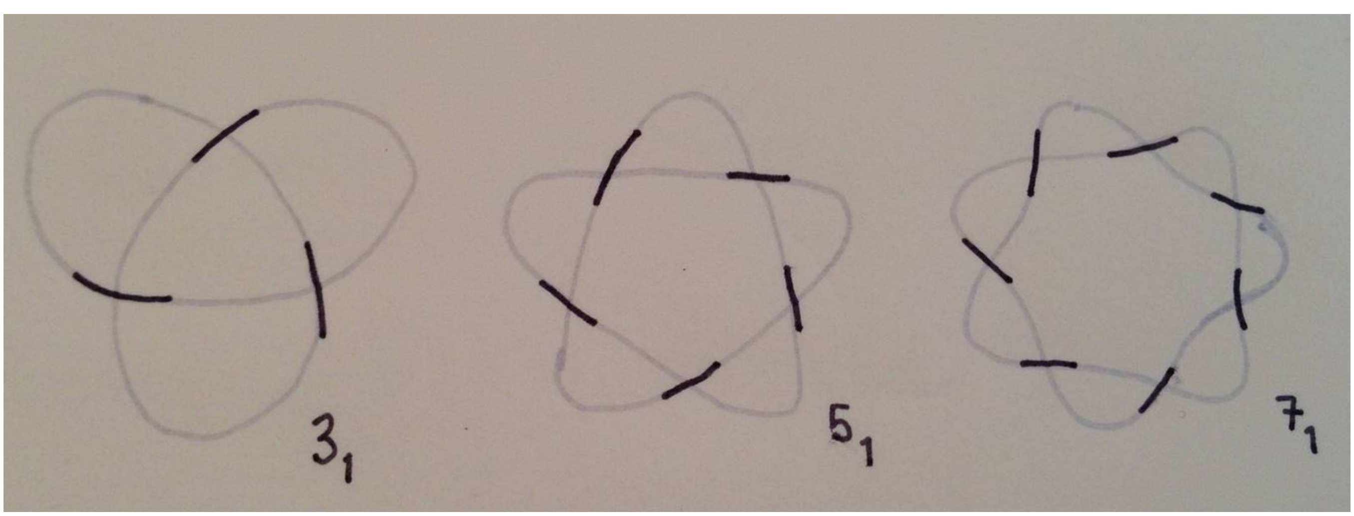
1 Braid Knots

Definition. *A* ***braid knot*** *is a knot with an odd minimal crossing number that is alternating, and is in the following form:*



A more intuitive way to think about braid knots is that they are knots constructed by repeatedly twisting strands around each other.

The following are examples of braid knots:



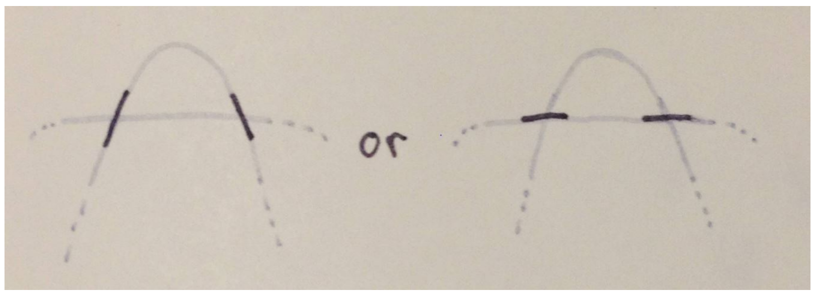
Theorem. *Given a braid knot with minimal crossing number n that is represented with minimal crossings, crossing changes will always take it to either a braid knot with minimal crossing number $\leq n$ or the unknot.*

Proof. Consider a general braid knot with minimal crossing number n.

Recall that a braid knot represented with minimal crossings is alternating.

Given a braid knot that has had crossing changes performed on it, we have the following two possibilities for pairs of crossings:

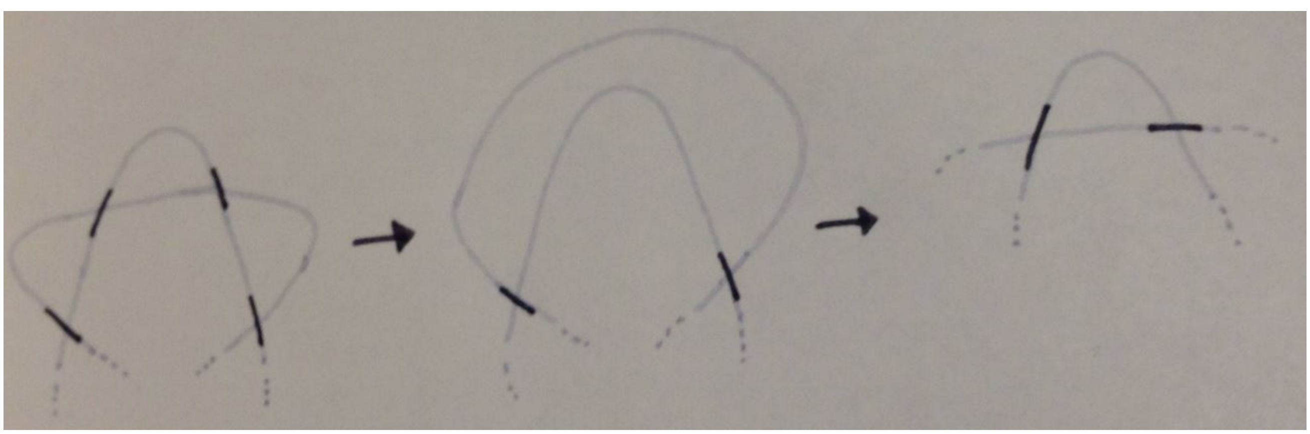
- The crossings are the same



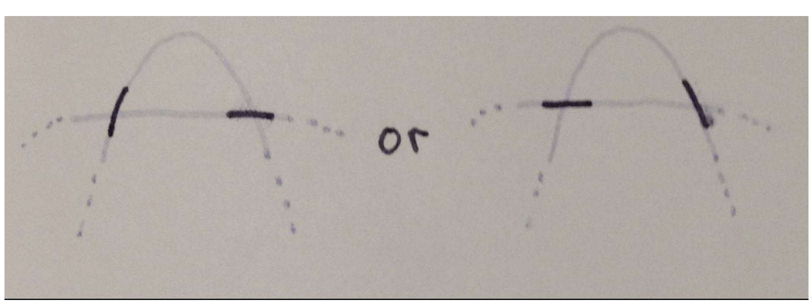
  We can see (by the second Reidemeister move) that we can pull the bottom strand out from under the top one, giving us this:

  and thereby reducing the minimal crossing number by 2.

  We note that this preserves the braid knot structure.



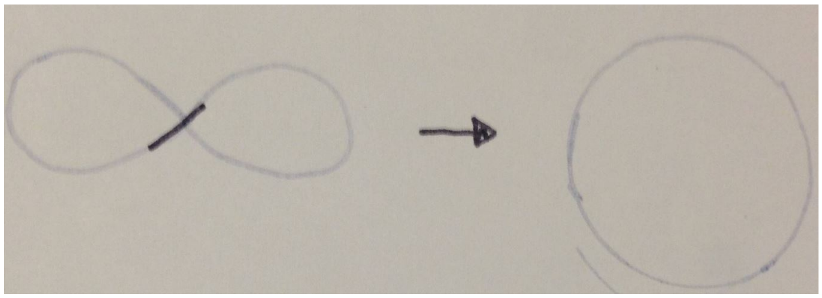
- The crossings are different



Since a braid knot is alternating, we can see that this both preserves the minimal crossing number and the braid knot structure.

Brief note: changing every crossing of a braid knot gives us a braid knot with the same minimal crossing number.

We can also see that if we reduce our braid knot to one with one crossing, this is the unknot.



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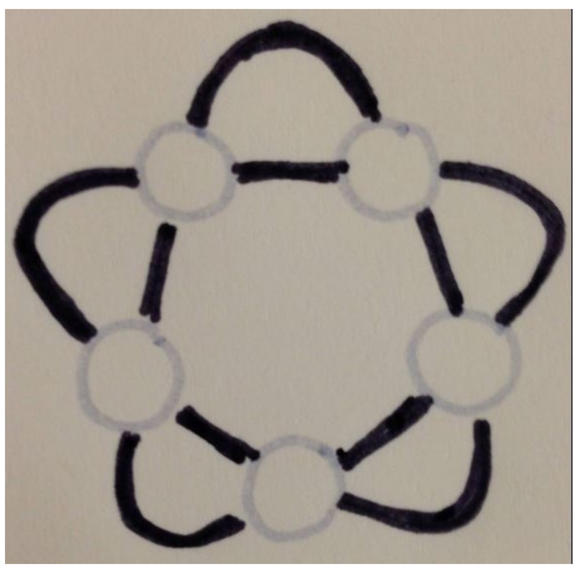
2 Crossing Game with Braid Knots

2.1 Strategy for 5_1

Recall that the crossing game with the 5_1 is with the following board:



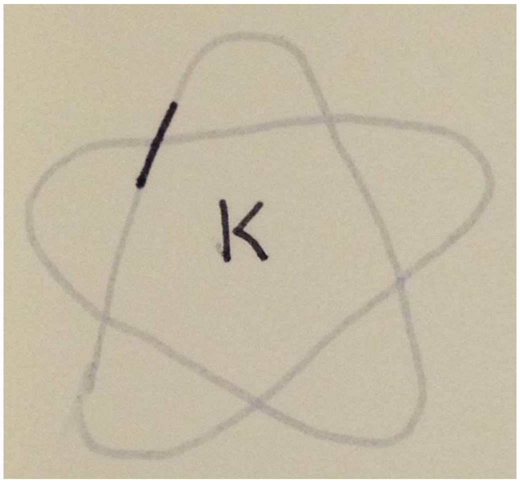
In the last write-up, we tried to find a winning strategy for this by brute force. Although there was a playable strategy, it was not very easy to see and we only considered when players had to make moves clockwise. We may want to know is if there is a simpler strategy and if it might be able to be generalized for the crossing game played with other braid knots.

Note: From now on, we will denote that a move is made by player U by placing a U inside our diagram for that move and that a move is made by player K by placing a K inside our diagram for that move.

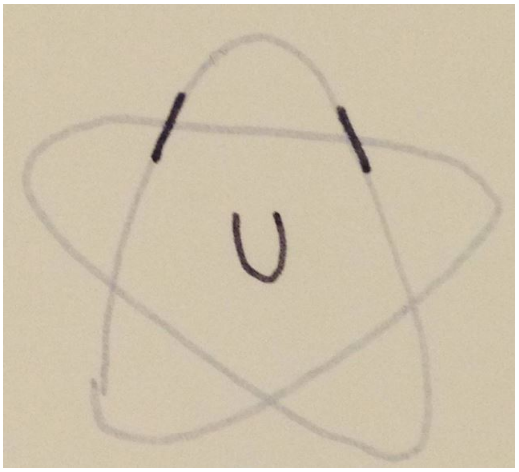
To find a winning strategy, we must consider the following two cases:

- K goes first

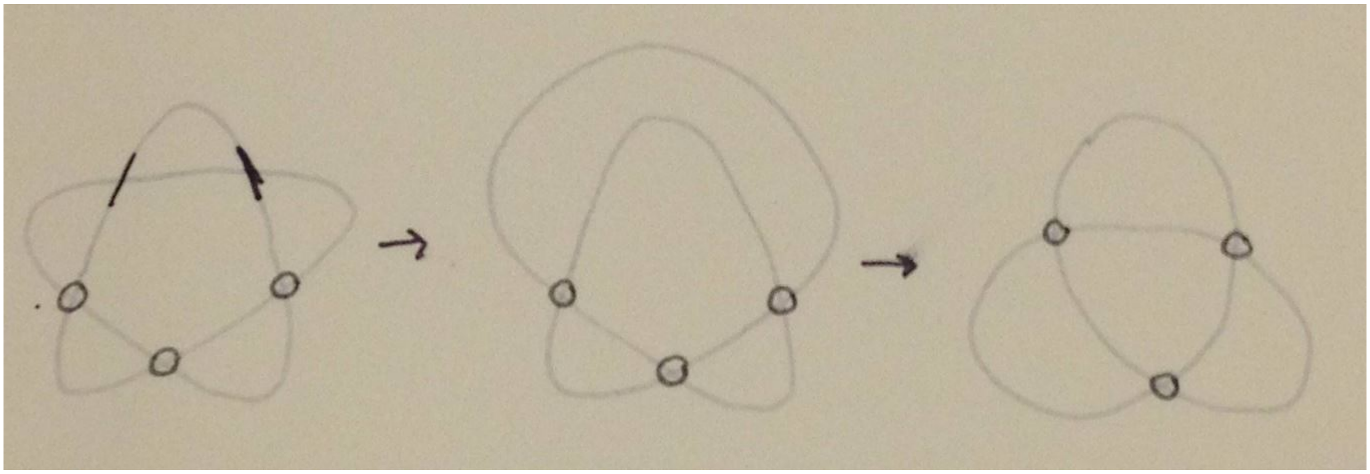
  The first move is arbitrary, therefore we can look at the following:



If player U forces the knot to have two consecutive overs, such as the following:



Then we can use the second Reidemeister move to separate the strands



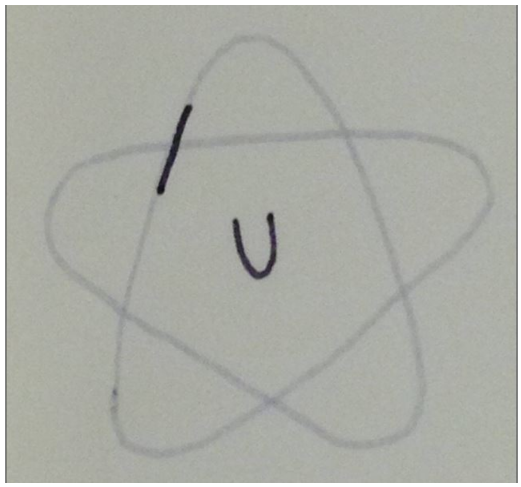
We can see that this gives us the trefoil. And since it is now Player K's turn, we can say that this game becomes the game for the trefoil with Player K making the first move.

Recall that no matter who goes first for the crossing game with the trefoil, Player U has a winning strategy.

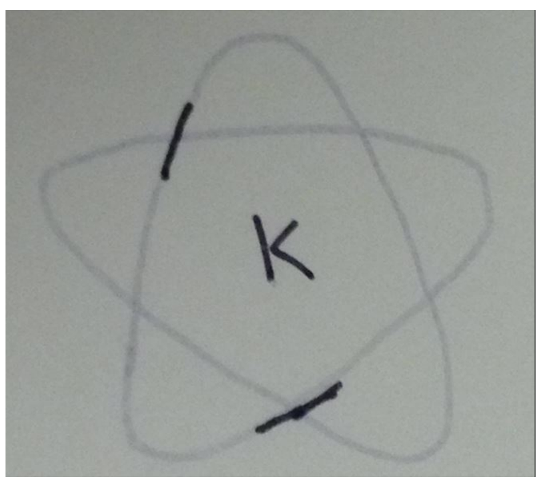
Therefore, if Player K goes first for the crossing game with the 5_1, Player U has a winning strategy.

- U goes first

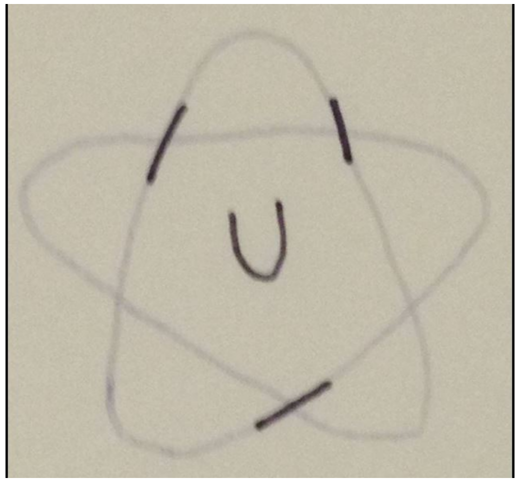
  The first move is arbitrary, therefore we can write the following:



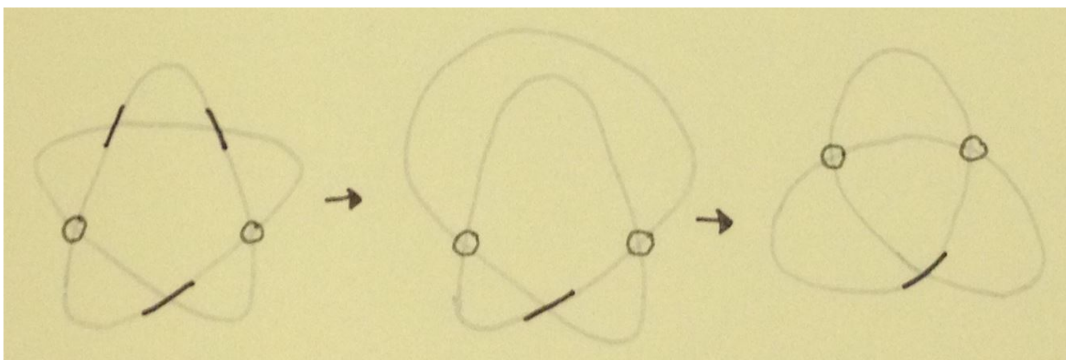
Player K can make any of the remaining moves. For this example, let Player K do the following:



If Player U forces the knot to have two consecutive overs, such as the following:



Then we can use the second Reidemeister move to separate the strands



We can see that this gives us the trefoil with one move already made. This is equivalent to the crossing game with the trefoil with U making the first move.

Recall that no matter who goes first for the crossing game with the trefoil, Player U has a winning strategy.

Therefore, if Player U goes first for the crossing game with the 5_1, Player U has a winning strategy.

Therefore, no matter who goes first for the crossing game with the 5_1, Player U always has a winning strategy.

2.2 Generalized Strategy for the Crossing Game with Braid Knots

Theorem. *Given a crossing game with any braid knot, regardless of who goes first, Player U has a winning strategy.*

Proof. By induction.

Recall that Player U has a winning strategy for the trefoil.

Therefore, in order to prove that Player U always has a winning strategy, we simply need to prove that Player U can always reduce the game to the trefoil.

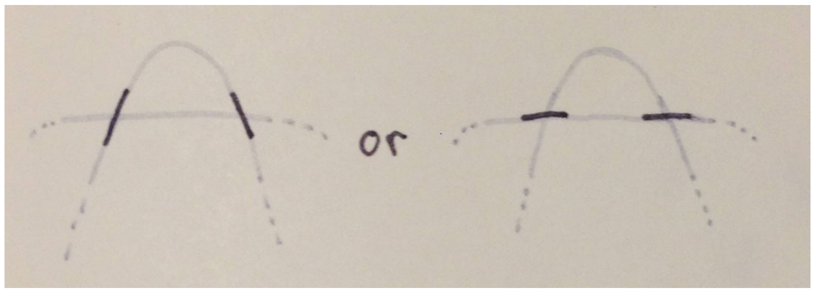
Recall that crossing changes will always take a braid knot with minimal crossing number n to either a braid knot with minimal crossing number $\leq n$ or the unknot.

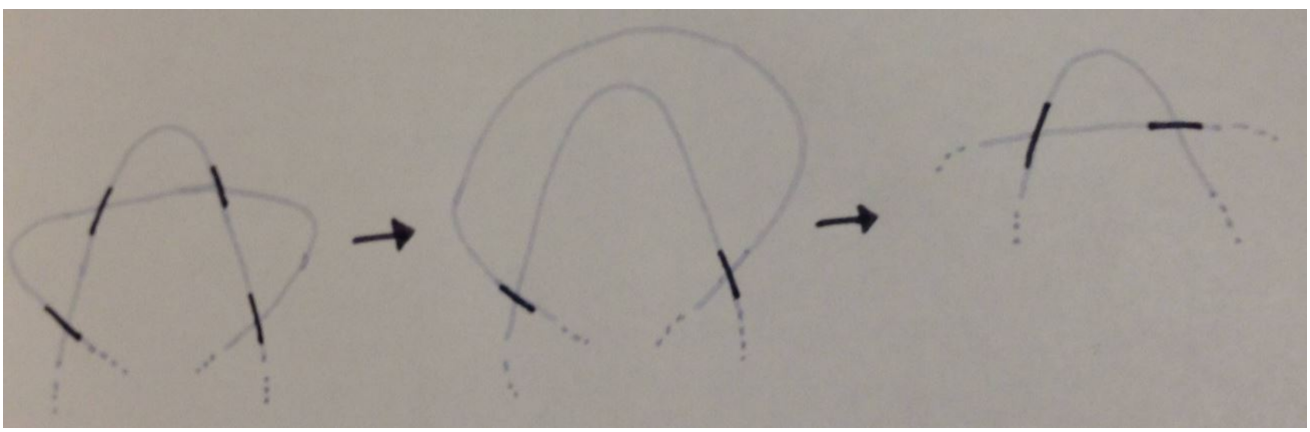
Suppose the crossing game is being played with a braid knot with minimal crossing number n. We have two possible cases:

- Player K goes first.

  The first move on braid knots is arbitrary.

  Player U makes the second move. Player U can make the knot have two consecutive overs,

  We can see that this gives us the board for a braid knot with minimal crossing number $n-1$ where Player K goes first.

- Player U goes first.

  The first move on braid knots is arbitrary.

  Player K makes the second move. And Player U makes the third move. Regardless of what move Player K makes, Player U can make the knot have two consecutive overs.

  Just as before, we can see that this gives us the board for a braid knot with minimal crossing number $n-1$, but with one move filled in (but this doesn't matter as first moves are arbitrary) where Player K goes second. In other words, for some braid knot with minimal crossing number n and Player U going first, Player U can always reduce this to a board with minimal crossing number $n-1$ and Player U going first.

We can see that for any n, we can repeat this process until we get to the trefoil. Since the Player U wins regardless of who goes first for the trefoil, Player U has a winning strategy for the crossing game with any braid knot. $\square$

3 References

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