Latin Squares
Week 3 Write-Up: Row Complete Latin Squares
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Let's say that you and three friends (We will call you Alice, Bob, Carol and Dave) want to have a very serious Super Smash Brothers[1] tournament. You want it to consist of one-on-one battles, and you want it to be as fair and balanced as possible. So you decide on the following rules:

- Each player will play each of the other players twice, once as Pikachu and once as Zelda.
- For simplicity, the tournament will start with the two people farthest left on the couch, and after every battle, players will pass the controller to the person on their right.
- After every three battles (when the controllers have reached the two people farthest right on the couch) the players will switch spots on the couch, so that each player is sitting in a different spot each time. This is so nobody has an advantage due to a better viewing angle to the TV.

So how do you make sure this format works? How do you know that it's possible to arrange yourselves on the couch in such a way that each unique match-up only happens once? This is where Latin squares can go to work for you! Consider the following Latin square with the symbols A, B, C, D (for Alice, Bob, Carol, and Dave).

A	B	C	D
B	D	A	C
C	A	D	B
D	C	B	A

This Latin square will plan your tournament! Here's how it works: Each row represents the order of people sitting on the couch, and each adjacent pair of elements represents a battle, with the person on the left playing Pikachu and the person on the right playing as Zelda. So in the first round, the order of people from left to right on the couch will be Alice, Bob, Carol, and Dave. First Alice will play Bob, Alice as Pikachu and Bob as Zelda, then they will pass their controllers to the right, and Bob will play Pikachu and Carol will play Zelda, then they will pass the controllers to the right, and Carol will be Pikachu and Dave will play Zelda. Then they will switch spots on the couch, and the order of people from left to right will be Bob, Dave, Alice and Carol.

So why does this Latin square make our tournament work? The fact that it is a Latin square ensures that nobody sits in the same couch spot twice (because each letter appears once in each column). As a matter of fact, each person will play two games in each of the middle couch spots, and one game in each of the far left and far right spots, so each person will get more time at a more direct angle to the TV. But how do we know that each match-up will only happen once? As a matter of fact, this isn't any ordinary Latin square. It is arranged so that each ordered pair of adjacent symbols appears once and only once. This ensures that every player will battle the other three players twice each, once as each character. This brings us to a definition:

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Author: Yian Huang

Complete Latin Squares are widely used in designing experiments. In this presentation I am going to introduce the formal definition of complete latin square. In order to introduce complete latin square, I first need to explain the concept of **row-complete latin square** and **column-complete latin square**.

To explain the concept of column and row complete latin square, I first need to clarify a small notation: $L_{i,j}$ This symbol represents the cell in the ith row, jth column of an $n \times n$ latin square. For example, in latin square A.

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

[1]This is a video game that all the kids are playing these days

$L_{1,2}$ represents the number 1 in 1st row, 2nd column.

$L_{5,5}$ represent the number 3 in 5th column, 5th row.

Q: What do $L_{2,2}$ and $L_{3,4}$ stand for?

1 Row complete latin square

First we are going to examine a example of row complete latin square B:

1	2	3	4
2	4	1	3
4	3	2	1
3	1	4	2

We notice that there are some patterns for each pair of number in rows:

| 1 | 2 | in line 1

| 1 | 3 | in line 2

| 1 | 4 | in line 4

And we find each number follows 1 only once in this latin square.

Actually till now we've figured out the definition of row complete latin square: For each ordered pair of elements in adjacent cells $L_{i,j}$ and $L_{i,j+1}$ in a row complete latin square, we have no 2 pairs that are the same in one latin square.

But there is one confusing thing,
For instance, in row complete Latin square B, we have
| 1 | 4 | in line 4 and

| 4 | 1 | in line 4

But we can still say the latin square is a row complete latin square because both ordered pairs (1,4) and (4,1) appear only once in B.

Q: Is latin square C row complete latin square?

1	2	3
2	3	1
3	1	2

2 Column Complete Latin Square

When we've already known row complete latin square, it is easy for us to comprehend the definition of column complete latin square, which, in particular, is: For each pair of $L_{i,j}$ and $L_{i+1,j}$ in column complete latin square, we have no 2 pairs that are the same in one latin square.

Q: Is B (above) a column complete latin square?

3 Complete Latin Square

The definition of complete latin square is also easy to understand now: A complete latin square is a square that is both row and column complete.

Q: Is B (above) a complete latin square? What about A?

4 What row complete and complete Latin squares exist?

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Here's a question that has not yet been completely answered: For what orders do row-complete and complete Latin squares exist? It has been known for a while that complete Latin squares exist for every even order, and they can be made by the following construction:

For even n, put the numbers 0 through $n-1$ in the first row in the following order: $0, 1, n-1, 2, n-2, \ldots n/2+1, n/2$. (For example, for $n=4$, we would have 0, 1, 3, 2). In each remaining empty cell, place the number in the cell directly above it plus 1 mod n. This is a row complete Latin square, and it can be made column complete by rearranging the first column so that it is in the same order as the first row. Here is an example for $n=4$: The first Latin square is row-complete, and the second Latin square is complete.

0	1	3	2
1	2	0	3
2	3	1	0
3	0	2	1

0	1	3	2
1	2	0	3
3	0	2	1
2	3	1	0

The proof that this Latin square is complete is left to the reader as an exercise. (Hint: Consider the differences between the elements in adjacent columns).

It was proven in 1998 by Jeff Higham that row-complete Latin squares exist for every composite order. No row-complete Latin square of prime order has yet been found.

Complete Latin squares exist for every even order, as well as of any order p^n, where p is a prime and $n > 2$. They also exist for order $3m$, where m is an odd powerful number (meaning that for every prime factor k of m, k^2 is a factor of m).

It is also known that if there is a sequenceable group of order n, then there is a complete Latin square of order n.

We say that a group is **sequenceable** if its elements can be arranged in a sequence $a_1, a_2, \ldots a_n$ so that the products $b_1 = a_1, b_2 = a_1a_2, \ldots, b_n = a_1a_2\ldots a_n$ are all distinct. For example, $\langle \mathbb{Z}/4\mathbb{Z}, + \rangle$ is sequenceable. We can arrange the elements in the order 0, 1, 2, 3 and the partial products $b_1 = 0, b_2 = 0+1 = 1, b_3 = 0+1+2 = 3, b_4 = 0+1+2+3 = 2$ are all distinct.

Claim: For any sequenceable group G, the matrix $c_{s,t} = b_s^{-1}b_t$, where $c_{s,t}$ means entry c in row s and column t, is a complete Latin square.

Proof. First, I will show that this is a Latin square. $b_1, \ldots b_n$ are all distinct elements of G by definition, and they all have a unique inverse. If the matrix were not a Latin square then there would exist some $b_i^{-1}b_j = b_i^{-1}b_k$, or some $b_i^{-1}b_j = b_m^{-1}b_j$, for some values of i, j, k and m. However, this implies that $j = k$, in the first instance, and that $i = m$ in the second instance. So each element appears exactly once in each row and column, and $c_{s,t}$ is a Latin square.

Now, we can show that it is horizontally complete. We proceed by contradiction. Assume $c_{u,v} = cx, y$ and $c_{u,v+1} = c_{x,y+1}$ are true for some values of u, v, x and y. I will show that $u = x$ and $v = y$. We have:

$$b_u^{-1}b_v = b_x^{-1}b_y$$

$$b_u^{-1}b_{v+1} = b_x^{-1}b_{y+1}$$

If we take the inverse of both sides of the first equation, we get $b_ub_v^{-1} = b_xb_y^{-1}$. We can combine this with the second equation:

$$(b_ub_v^{-1})(b_u^{-1}b_{v+1}) = (b_xb_y^{-1})(b_x^{-1}b_{y+1})$$

$$b_v^{-1} b_{v+1} = b_y^{-1} b_{y+1}$$

This means that

$$(a_1 a_2 \dots a_v)^{-1}(a_1 a_2 \dots a_v a_{v+1}) = (a_1 a_2 \dots a_y)^{-1}(a_1 a_2 \dots a_y a_{y+1})$$

$$a_{v+1} = a_{y+1}$$

This means that $v = y$. This turns our first equation into $b_u^{-1} b_v = b_x^{-1} b_v$. This gives us that $u = x$. So our Latin square is horizontally complete.The proof of vertical completeness is nearly identical. □

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Sources: Latin Squares and their Applications, J. Denes & A.D. Keedwell

"Row Complete Latin Squares", University of Surrey

"Row Complete Latin Squares of Every Composite Order Exist", Jeff Higham, Journal of Combinatorial Designs

"Complete latin squares of order p^n exist for odd primes p and n¿2", Chengde Wang, Discrete Mathematics

"New Complete Latin Squares of Odd Order", M.A. Ollis, European Journal of Combinatorics

"Sequences in Groups With Distinct Partial Products", Basil Gordon, Pacific Journal of Mathematics