Latin Squares Week 6: Coy Completeness in Latin Cubes

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1 Introduction

In this presentation we are going to introduce $\varphi(B)$ completeness of latin cubes. I'm going to introduce the formal definition of $\varphi(B)$ completeness and the number of possible $\varphi(B)$ squares in a latin cube. To be specific, $\varphi(B)$ completeness is catalogued as one way $\varphi(B)$ complete, two way $\varphi(B)$ complete and three way $\varphi(B)$ complete in a latin cube.

To explain the concept of $\varphi(B)$, I first need to recall the small notation introduced in the last class. $\varphi_{i,j,k}$
This symbol represents the cell in ith row, jth column and k th layer of an $n \times n$ latin cube.

Next we are going to introduce one way $\varphi(B)$ completeness in a latin cube.

2 One way $\varphi(B)$ completeness in a latin cube

First we are going to examine a example of a row complete latin square B, which can be considered as the first layer of a latin cube, therefore, k=1:

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

We can look at some 2x2 subsquares within the square:

1	2
2	4

starts in position $\varphi_{1,1,1}$.

1	3
4	2

starts in position $\varphi_{2,3,1}$

1	4
3	2

starts in position $\varphi_{3,2,1}$

And we find of this squares only appears once in the latin cube A:

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

.....layer 1.

2	3	4	1
3	1	2	4
4	2	1	3
1	4	3	2

..... layer 2.

3	4	1	2
4	2	3	1
1	3	2	4
2	1	4	3

..... layer 3.

4	1	2	3
1	3	4	2
2	4	3	1
3	2	1	4

.... layer 4.

In this cube no 2x2 subsquare appears more than once.

We say that a Latin cube φ is *one − way* $\varphi(B)$ *complete* if 2x2 squares:

$\varphi_{i,j,k}$	$\varphi_{i+1,j,k}$
$\varphi_{i,j+1,k}$	$\varphi_{i+1,j+1,k}$

appear only once in the cube.

There is nothing special about i and j here, in other words, a cube in which each of the following 2x2 squares appears only once would also be $\varphi(B)$ complete:

$\varphi_{i,j,k}$	$\varphi_{i+1,j,k}$
$\varphi_{i,1,k+1}$	$\varphi_{i+1,j,k+1}$

Same with these:

$\varphi_{i,j,k}$	$\varphi_{i,j+1,}$
$\varphi_{i,j,k+1}$	$\varphi_{i,j+1,k+1}$

3 Two way $\varphi(B)$ complete

When we've already known one way $\varphi(B)$ complete latin cube, it is easy for us to comprehend the definition of a 2-way $\varphi(B)$ complete Latin cube, which is: Every 2x2 square of the following two types appears only once in a Latin cube φ:

$\varphi_{i,j,k}$	$\varphi_{i+1,j,k}$
$\varphi_{i,1,k+1}$	$\varphi_{i+1,j,k+1}$

and

$\varphi_{i,j,k}$	$\varphi_{i,j+1,}$
$\varphi_{i,j,k+1}$	$\varphi_{i,j+1,k+1}$

Alternatively, we could look at every 2x2 square of the following two types:

$\varphi_{i,j,k}$	$\varphi_{i+1,j,k}$
$\varphi_{i,1,k+1}$	$\varphi_{i+1,j,k+1}$

and

$\varphi_{i,j,k}$	$\varphi_{i+1,j,k}$
$\varphi_{i,j+1,k}$	$\varphi_{i+1,j+1,k}$

Or every 2x2 square of the following two types:

$\varphi_{i,j,k}$	$\varphi_{i,j+1,}$
$\varphi_{i,j,k+1}$	$\varphi_{i,j+1,k+1}$

and

$\varphi_{i,j,k}$	$\varphi_{i+1,j,k}$
$\varphi_{i,j+1,k}$	$\varphi_{i+1,j+1,k}$

Q: Is latin Cube A (from the previous page) two way complete? If it is, in which 2 ways it is complete? if not, explain the reason.

4 Three way $\varphi(B)$ complete

A 3 way $\varphi(B)$ complete latin cube is a cube such that each of the following three types of 2x2 squares appears only once:

$\varphi_{i,j,k}$	$\varphi_{i,j+1,}$
$\varphi_{i,j,k+1}$	$\varphi_{i,j+1,k+1}$

and

$\varphi_{i,j,k}$	$\varphi_{i+1,j,k}$
$\varphi_{i,j+1,k}$	$\varphi_{i+1,j+1,k}$

and

$\varphi_{i,j,k}$	$\varphi_{i+1,j,k}$
$\varphi_{i,1,k+1}$	$\varphi_{i+1,j,k+1}$

So no 2x2 square appears more than once in the Latin cube, in any of the three possible sets of n faces.

In addition, I refer to a 2x2 square within a Latin cube as a $\varphi(B)$ square.

Q: Is the following Latin cube a three way $\varphi(B)$ complete latin cube?

1	2
2	1

2	1
1	2

5 Computing the number of possible $\varphi(B)$ squares in a latin cube

First lets take a look at latin cube

1	2
2	1

2	1
1	2

It very intuitive that only 2 kinds of $\varphi(B)$ square are in this cube.

then lets look at 4X4 latin cube like this one.

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

.....first layer.

2	3	4	1
3	1	2	4
4	2	1	3
1	4	3	2

.....second layer.

3	4	1	2
4	2	3	1
1	3	2	4
2	1	4	3

.....second layer.

4	1	2	3
1	3	4	2
2	4	3	1
3	2	1	4

.....forth layer.

In this one way complete latin cube, we can see there are possible patterns like

1	2
2	4

and

3	4
1	3

....

To calculate the possible patterns that are not that intuitive. We need to introduce a way to compute the number of $\varphi(B)$ squares in a nxn latin cube.

First we look at n=4 case:

In this empty square, which is a part of a latin cube, we first need to find out how many possible squares exist.

The top left cell can either be 1, 2, 3 or 4.

We can, for instance chose 2 in this cell. Then the latin square looks like:

2	?
?	?

$Possibilites = 4$	$Possibilities =?$
$Possibilities =?$	$Possibilities =?$

For the 2nd empty cell on the top right, according to the rules of latin squares, we have 3 possible numbers to fill in the cell.
For instance, if we chose a 2 in the first cabin, the next column we only have 3 possible numbers for the cell which are either 1 or 4 or 3. We can, for instance chose 3 in this cabin. Then the latin square looks like:

2	3
?	?

The possibility square (representing the number of possible choices for each cell) looks like

$Possibilities = 4$	$Possibilities = 3$
$Possibilities =?$	$Possibilities =?$

For the bottom left cell, we also have 3 possible numbers, which are 1 or 4 or 3.
If we also chose 3 in this case.
we can construct a latin square like

2	3
3	?

And the fourth column has 3 possible numbers which are 4 and 1 and 2

2	3
3	$4or1or2$

The possibility square looks like:

$Possibilities = 4$	$Possibilities = 3$
$Possibilities = 1$	$Possibilities = 3$

We say that the bottom left square has one possibility because we are considering the case where it has the same number as the bottom left square. We now consider the other case:
If we also chose 1 or 2 in this case.
we can construct a latin square like

2	3
1	?

And the bottom right cell only has 2 possible numbers which are 4 and 2

2	3
1	$4or2$

$Possibilities = 4$	$Possibilities = 3$
$Possibilities = 2$	$Possibilities = 2$

Eventually we construct 2 possibility cases which are

$Posbility = 4$	$Posbility = 3$
$Posbility = 2$	$Posbility = 2$

and

$Posbility = 4$	$Posbility = 3$
$Posbility = 1$	$Posbility = 3$

Therefore in this case the total possibility of looks for a $\varphi(B)$ in a 4x4 latin cube is:
$4 \times 3 \times 1 \times 3 + 4 \times 3 \times 2 \times 2 = 84$
What if we look at a more general nxn latin square? we can also construct possibility squares for $\varphi(B)$ squares using above method. To be specific.
For the top left cell, all numbers from 1 through n are possible.
We can, for instance, chose a in this cell. Then the latin square looks like:

a	?
?	?

And the possibility square looks like:

$Possibility = n$	$Possibility =?$
$Possibility =?$	$Possibility =?$

For the top right cell, according to the rules of latin squares, we have $n-1$ possible numbers for the cell.
For instance, if we chose an a in the first cell, the top right cell we only have $n-1$ possible numbers can be in the cell. We can, for instance chose b in this cell. Then the latin square looks like:

a	b
?	?

The possibility square like

$Possibility = n$	$Possibility = n-1$
$Possibility =?$	$Possibility =?$

For the bottom left cell, we also have $n-1$ possible numbers which are all numbers 1 through n except a
If we also chose b in this case.
we can construct a latin square like

a	b
b	?

And the forth column has $n-1$ possible numbers which all numbers except b. The possibility square looks like:

a	b
b	$notb$

$Posbility = n$	$Posbility = n-1$
$Posbility = 1$	$Posbility = n-1$

If we chose a number other than a and b in this case, like c, we have n-2 choices.
we can construct a latin square like

a	b
c	?

And the forth column only has n-2 possible numbers which not b and c

a	b
c	$notborc$

$Posbility = n$	$Posbility = n-1$
$Posbility = n-2$	$Posbility = n-2$

In total,the number of possible $\varphi(B)$ squares for some order n is:
$n(n-1)(n-1) \times 1 + n(n-1)(n-2)(n-2) = n(n-1)(n-1+(n-2)^2)$
.

Now we can consider the number of 2x2 squares that actually exist in a Latin cube. Each layer of a cube has $n-1$ pairs of adjacent rows, and $n-1$ pairs of adjacent columns, so there are $(n-1)^2$ 2x2 squares in each layer. For one-way completeness, we are considering n layers, so there are $n(n-1)^2$ 2x2 squares within the cube. For two-way completeness, we are considering 2n layers, so there are $2n(n-1)^2$ 2x2 squares within the cube. For three-way completeness, we consider $3n(n-1)^2$ 2x2 squares in the cube. These bounds, when compared to the formula above, tell us that a Latin cube of order 3 or smaller cannot be 2 or 3-way $\varphi(B)$ complete, and a cube of order 4 cannot be 3-way $\varphi(B)$ complete.
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A question you might ask is, "For what orders are $\varphi(B)$ complete Latin cubes known to exist?" Due to the very new nature of this topic, not much is yet known in answer to this question. However, we do know the following:

Claim: There exists a one-way $\varphi(B)$ complete Latin cube of every even order.

Proof. The reader may remember the following construction for a row and column complete Latin square for every even order:

For even n, put the numbers 0 through $n-1$ in the first row in the following order: $0, 1, n-1, 2, n-2, \ldots n/2+1, n/2$. (For example, for $n = 4$, we would have 0, 1, 3, 2). In each remaining empty cell, place the number in the cell directly above it plus 1 mod n. This is a row complete Latin square, and it can be made column complete by rearranging the first column so that it is in the same order as the first row.

Clearly, since this square is row complete, it is also $\varphi(B)$ complete. If each pair only appears once, then each 2x2 square will certainly only appear once. We can extend this construction to a one-way $\varphi(B)$ complete Latin cube L as follows: Make the first layer of the Latin cube the complete Latin square from the construction above. To make subsequent layers, take each empty cell located at some (i, j, k) (row, column and aisle) and place in it the number $L_{i,j,k-1}$ plus 1 mod n. This Latin cube is now one-way $\varphi(B)$ complete in one direction–that is, for all of the (i, j) squares in the cube. To make it one-way $\varphi(B)$ complete in all three directions, rearrange the layers so that the top left aisle (the one consisting of $L_{1,1,1}, L_{1,1,2} \ldots L1, 1, n$) is in the same order as the first row of the first layer. For example, our 4x4x4 one-way $\varphi(B)$ complete Latin cube can be constructed from these methods:

0	1	3	2
1	2	0	3
3	0	2	1
2	3	1	0

1	2	0	3
2	3	1	0
0	1	3	2
3	0	2	1

3	0	2	1
0	1	3	2
2	3	1	0
1	2	0	3

2	3	1	0
3	0	2	1
1	2	0	3
0	1	3	2

You may recall that we proved the construction for a complete Latin square by examining the differences between columns and rows. We showed that each pair of adjacent columns had a unique difference between its elements. Additionally, each pair of adjacent rows has a unique difference between its elements, because the difference between adjacent rows was 1 until we rearranged the rows so that the first column was in the same order as the first row. Each 2x2 square, S, within the complete Latin square contains a unique set of pairs of rows and columns. This means that only a square in S's position could have the unique ordered pair (DR_S, DC_S), where DR_S is the difference between elements in S's rows, and DC_S is the difference between elements in S's columns. So when we construct the next layers by adding 1 to all of the symbols in the complete Latin square, we preserve the differences between rows and columns. So the square in each layer in S's position will have the same difference between rows and columns, but will be different because we are adding 1 to the symbols. So each 2x2 square will appear at most once through the n layers, and the cube is 1-way $\varphi(B)$ complete. When we rearrange the layers, it ensures all three sets of n layers are one-way complete. (In fact, each set of n layers will be the same four Latin squares in the same order). $\square$

So what else do we know? As we showed earlier, there are not enough possible 2x2 squares for a 2x2 or 3x3 Latin cube to be more than 1-way $\varphi(B)$ complete. The only 2x2 Latin cube is 1-way $\varphi(B)$ complete:

1	2
2	1

2	1
1	2

We can determine through case-checking that there is no $\varphi(B)$ complete Latin cube of order 3. In fact, there is not even a 3x3 $\varphi(B)$ complete Latin square. We can assume without loss of generality that the first row of the square is 1 2 3. Here are the two possible squares:

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

Both of these Latin squares have repeated 2x2 squares, so neither of them can be a part of a $\varphi(B)$ complete Latin cube. So there is no $\varphi(B)$ complete Latin cube of order 3.

For order 4, we have shown already that there exists a 1-way $\varphi(B)$ complete Latin cube, and that there are not enough 2x2 Latin squares for there to be a 3-way $\varphi(B)$ complete cube. A search is currently under way for a 2-way complete cube. One has not yet been found, and it is known that there does not exist one with the complete Latin square as a layer.

For order 5 and above, aside from there existing a 1-way $\varphi(B)$ complete Latin cube of every even order, this question is open.