Latin Squares Week 9: $\varphi(B)$ Completeness and Sequenceable Groups

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This week, we discuss the emerging field of $\varphi(B)$ complete Latin cubes, currently being studied by mathematicians around the world[1]. Then, we discuss the somewhat related topic of sequenceable groups and how they relate to Latin squares.

First, let's review a few definitions:

We say that a Latin cube φ, where $\varphi_{i,j,k}$ is the entry in the ith row, jth column, and kth layer, is $one - way$ $\varphi(B)$ *complete* if each ordered quadruple
$(\varphi_{i,j,k}, \varphi_{i+1,j,k}, \varphi_{i,j+1,k}, \varphi_{i+1,j+1,k})$
appears only once in the cube.

A Latin square L is *row-complete* if each ordered pair of adjacent symbols $(L_{i,j}, L_{i,j+1})$ appears only once in the square.

L is column complete if each ordered pair $(L_{i,j}, L_{i+1,j})$ appears only once in the square.

I present the following new result about one-way $\varphi(B)$ complete Latin cubes:

Theorem: There exists a one-way $\varphi(B)$ complete Latin cube of every composite order.

Proof. To show this, we use a result by Jeff Higham, who proved that there exists a row-complete Latin square of every composite order. The proof of this is beyond the scope of this write-up, but the paper is listed in our sources.

I will show that you can take any row-complete Latin square and use it to make a one-way $\varphi(B)$ complete Latin cube. So take any row-complete Latin square. Clearly the square is $\varphi(B)$ complete, because since each ordered pair of adjacent symbols appears only once, each 2x2 square will appear only once. Label each of the rows of the square $R_1, R_2, \dots R_n$. Now flip the square, if necessary, so that it is column-complete, and re-label all the symbols to $R_1, R_2, \dots R_n$, etc. Now each column of this square represents a layer of a Latin cube, using the rows $(R_1, R_2,$, etc) in the order they appear in the column.

Here's an example when $n = 4$:

We take a row-complete Latin square of order 4, like this one:

0	1	3	2
1	2	0	3
2	3	1	0
3	0	2	1

We label the rows: 0132 is R_1, 1203 is R_2, 2310 is R_3, and 3021 is R_4. Now we flip the Latin square

[1]By which I mean myself.

diagonally to get a column complete Latin square:

0	1	2	3
1	2	3	0
3	0	1	2
2	3	0	1

We relabel the symbols to R_1, R_2, R_3, and R_4:

R_1	R_2	R_3	R_4
R_2	R_3	R_4	R_1
R_4	R_1	R_2	R_3
R_3	R_4	R_1	R_2

This gives us the following one-way $\varphi(B)$ complete Latin cube:

0	1	3	2
1	2	0	3
3	0	2	1
2	3	1	0

1	2	0	3
2	3	1	0
0	1	3	2
3	0	2	1

2	3	1	0
3	0	2	1
1	2	0	3
0	1	3	2

3	0	2	1
0	1	3	2
2	3	1	0
1	2	0	3

So why does this construction always work? As stated before, each ordered pair appears only once in any set of the four rows. Since the Latin square with the symbols $R_1, \ldots, R_n$ is column complete, each row appears above each other row exactly once, and below each other row exactly once. So for any given pair of adjacent symbols, each pair that appears directly below it will be different, and each pair that appears directly above it will be different. So each 2x2 square will appear at most once in the four layers. $\square$

As a review of what is open in this area, we do not know if there exist 1-way $\varphi(B)$ complete Latin cubes of prime order.

We do not know if there exist 2-way $\varphi(B)$ complete Latin cubes of order 4 or above.

We do not know if there exist 3-way $\varphi(B)$ complete Latin cubes of order 5 or above.

We still have not found a 2-way or 3-way $\varphi(B)$ complete Latin cube, but the search continues. A method like the one above, consisting of rearranging rows, would not work, because each row can only appear above $n-1$ other rows, so that only works across n layers. Any construction, if it existed, would likely be pretty complex.

Sequenceable Groups

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A group is sequenceable if there is some way to arrange the group's elements $\{a_1, a_2, ..., a_n\}$ such that the sequence of products $\{a_1, a_1a_2, ..., a_1a_2...a_n\}$ is also some arrangement of the groups elements. For ease, we will let $b_i = a_1a_2...a_i$.

For an example of a sequenceable groub, the group $< \mathbb{Z}/4\mathbb{Z}, + >$, is sequenceable:

$\{a_1, a_2, ..., a_n\}$
$= \{0, 1, 2, 3\}$

$\{a_1, a_1a_2, ..., a_1a_2...a_n\}$
$= \{b_1, b_2, ..., b_n\}$
$= \{0, 0+1, 0+1+2, 0+1+2+3\}$
$= \{0, 1, 3, 2\}$

Because both of these sets are some arrangement of our group's elements, our group is sequenceable.

We can use this concept to relate to complete Latin Squares.
Claim:
Let G be a sequenceable group of order n, and $\{a_1, a_2, ..., a_n\}$ be some ordering of G which produces some $\{b_1, b_2, ..., b_n\}$ that is also an ordering of G. Then we can form a complete Latin Square L, where $L = (l_{ij})$, where $(l_{ij} = b_i^{-1} b_j)$ for $1 \leq i, j \leq n$.

Proof. Firstly, We must show that this creates an $n \times n$ Latin Square.
We can show this easily. Assume $b_i^{-1} b_j = b_i^{-1} b_k$, where $1 \leq k \leq n$. We want to show that if this is true, then $j = k$.
$b_i^{-1} b_j = b_i^{-1} b_k$
$\Rightarrow b_j = b_k$
$\Rightarrow j = k$
A similar argument applies for columns:
$b_i^{-1} b_j = b_h^{-1} b_j$
$\Rightarrow b_h^{-1} = b_i^{-1}$
$\Rightarrow i = h$

Now, we have shown that we have a Latin square. We must now show that our square is row-complete:
Suppose $(l_{ij}, l_{i,j+1}) = (l_{hk}, l_{h,k+1})$
Thus, we have $l_{ij} = l_{hk}$, and $l_{i,j+1} = l_{h,k+1}$
These can be written as:
Equation 1: $b_i^{-1} b_j = b_h^{-1} b_k$
Equation 2: $b_i^{-1} b_{j+1} = b_h^{-1} b_{k+1}$

We invert both sides of Eqn 1 to get $b_j^{-1} b_i = b_k^{-1} b_h$. Combining this with Eqn 2 gives us the following:
(Eqn 1)$^{-1}$(Eqn 2)
$\Rightarrow b_j^{-1} b_i b_i^{-1} b_{j+1} = b_k^{-1} b_h b_h^{-1} b_{k+1}$
$\Rightarrow b_j^{-1} b_{j+1} = b_k^{-1} b_{k+1}$
$\Rightarrow a_{j+1} = a_{k+1}$
$\Rightarrow j = k$
If we substitute this result into Eqn 1, we get:
$b_i^{-1} b_j = b_h^{-1} b_j$
$\Rightarrow b_i^{-1} = b_h^{-1}$
$\Rightarrow i = h$

Thus, we have proven that if $l_{ij} = l_{hk}$ and $l_{i,j+1} = l_{h,k+1}$, then $j = k$ and $i = h$, thus proving that our Latin Square is row-complete.
The proof to show it is column-complete is very similar, and uses the same methods, so we omit this for brevity.

Now, we have proven that any sequenceable group of order n gives us a construction for a complete latin square of order n! $\square$

Sources: "Row Complete Latin Squares of Every Composite Order Exist", Jeff Higham, Journal of Combinatorial Designs, 1998

"Sequences in Groups With Distinct Partial Products", Basil Gordon, Pacific Journal of Mathematics, 1961