Knot Theory Week 6: Link Smoothing Game (2)

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We introduced the Link Smoothing Game last time. In general, there are actually two representations in the game: Link shadow and checkerboard coloring.

1 Link Shadow

Link Shadow is a connected link diagram where under- and over-strand information is unspecified at the crossings, which we used to play the game last time.
A link shadow D is in one of four outcome classes.
N-position, provided that the next player to move (or first player) can guarantee a win;
P-position, provided that the previous player (or second player) can guarantee a win;
L-position, provided that L can guarantee a win regardless of moving first or second;
K-position, provided that K can guarantee a win regardless of moving first or second.
The lemma "if L plays last, then L wins" implies there are no K-position link shadows.
Thus, as K wins seem rare, we wish to identify all N and P-position shadows. We shall assume K moves last and describe the winning strategy for K with some knowledge in graph theory.
Before we use planar graphs to figure out the winning strategy, we should know some concepts in graph theory here.
Face: For any connected planar graph G, we can define a **face** of G to be a connected region of R^2 whose boundary is given by the edges of G.
Loop: In graph theory, a loop is an edge that connects a vertex to itself.
Cut-edge: An edge $e \in G$ is called a cut-edge if deleting e from G increases the number of connected components of G.
Leaf: a leaf is a vertex whose degree is 1.

2 Checkerboard Coloring

There is actually another representation of the diagram: The checkerboard coloring (Figure 2). The **checkerboard coloring** is to assign black and white to different regions (faces), such that if two faces share the same edge, then their colors are different. We can find out the winning strategies by capitalizing on the relationship between link shadows and the planar graphs associated to their checkerboard colorings.

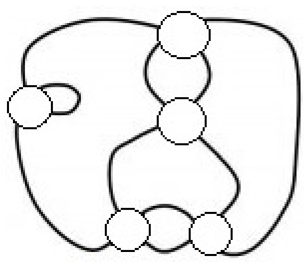


Figure 1

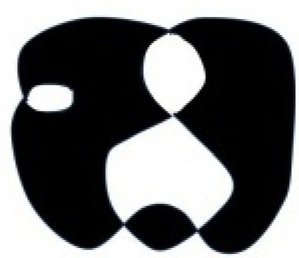


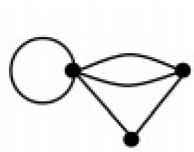
Figure 2

The shadow of a knot or link can be represented using a planar graph. This representation gives us tremendous insight into the game strategy.

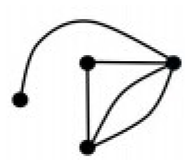
3 Black graph and white graph

How to consturct the black graph and the white graph?

Vertices in the graph are in one-to-one correspondence with the black regions of the checkerboard coloring and there is an edge between vertices for each precrossing between the corresponding black regions. The graph of a shadow may contain **loops** or multiple edges between a given vertex pair. We call the graph of the shadow **the black graph** and its dual we call **the white graph**.



Black Graph



White Graph

The two graph above are the black graph and the white graph of the figure 2 shadow. By observation, we find that a smoothing within a shadow corresponds either to (1) **an edge contraction** or (2) **an edge deletion** in the in the corresponding graph.

More precisely, an edge deletion in the black graph corresponds to a smoothing that

eliminates an adjacency between black region(s) of the checkerboard coloring. This smoothing in turn corresponds to joining white region(s) in the checkerboard coloring and thus contracting the corresponding edge in the white graph that is dual to the removed edge in the black graph.
Similarly, a smoothing that corresponds to an edge contraction in the black graph gives rise to the deletion of the dual edge in the white graph. We observe that the shadow is connected if and only if both the black and white graphs are connected.

4 Using graphs to find winning strategies for L

Proposition 1. *Suppose L plays first in a game on link shadow diagram D corresponding to an embedding of connected planar graph G.*
Then L has a winning strategy if G contains any of the following.
1. a cut edge, i.e. an edge e such that G e is disconnected,
2. a loop,
3. a pair of vertices joined by more than two edges.

Proof. We first note that L wins if a move is made such that either G or its dual, G^* , becomes disconnected.
(1) Deleting a cut edge disconnects the graph, yielding an L win.
(2) We note that a loop in G corresponds to a leaf in G^* , which is a particular type of cut edge. Thus, L can disconnect the graph on her first move.
(3) If a pair of vertices is joined by more than two edges, then L can contract one of the edges to produce two or more loops. At least one loop will remain in Ls second turn, so L has a winning strategy as in (2). □

5 Using graphs to find winning strategies for K

We will see that defensive strategies for Knot can be found for shadows whose graphs contain certain pairings of edges.

5.1 N-edge blowout

Definition 4. Let G be a finite planar graph with a fixed planar embedding **D** and let v be a vertex of G. An n-edge blowout of G at v is a planar graph obtained from D by selecting a cyclic partition of the edges emanating from v into two subsets of edges, "splitting" the vertex v into two vertices v' and v'' so that each of the two edge subsets are connected to v' or v'', and then adding n new edges to the diagram between v' and v'' in any way that results in a planar graph.

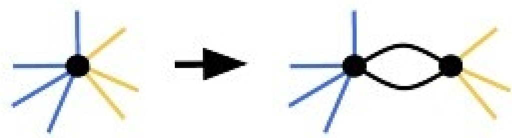


Figure 3 – The local picture of a 2-edge blowout at v. On the left, the partition of edges is denoted by the edge colors pink and blue. On the right is the blowout corresponding to this partition.

Proposition 2. Suppose shadow diagram D corresponds to a graph G. Let G be a graph that can be constructed from a single vertex by a sequence of 2-edge blowouts. Then K has a winning strategy in a game on D iff K plays second. Thus, D is a P-position game.

Proof. Let us use the notation $B_{e_i,e_{i+1}}$ to refer to a 2-edge blowout on a graph G that introduces the two new edges e_i and e_{i+1}. We note that deleting one of e_i, e_{i+1} and contracting the other is the inverse operation of $B_{e_i,e_{i+1}}$, returning us to our original graph G.

We proceed with our proof by induction. Consider the simplest non-trivial graph produced by a sequence of 2-edge blowout B_{e_1,e_2}. If L deletes (contracts) one edge in the graph, then K can contract (delete) the other edge to produce a single vertex.

(We should note that contracting an edge in a black graph corresponds to deleting an edge in the white graph, and the white graph of a 2-cycle is also a 2-cycle, so a single vertex is produced by this sequence of moves in the white graph as well.)

We conclude tht K has a winning strategy in this game, if K moves second.

Assume that K has a winning strategy on any graph produced by n 2-edge blowouts, B_{e_1,e_2}, B_{e_3,e_4}, ... $B_{e_{2n-1},e_{2n}}$. Considering a graph resulting from n+1 2-edge blowouts:

$$B_{e_1,e_2},\ B_{e_3,e_4},\ \dots\ B_{e_{2n-1},e_{2n}},\ B_{e_{2n+1},e_{2n+2}}$$

If L moves on e_{2n+1} or e_{2n+2}, then K can make the complimentary move on the other edge in the pair to reduce the game to the one produced by B_{e_1,e_2}, B_{e_3,e_4}, ... $B_{e_{2n-1},e_{2n}}$. Thus, if L's first move is in one of the two edges produced by the final 2-edge blow out, then K has a winning strategy.

The parity of the game allows K to force L to move first in the pair e_{2n+1},e_{2n+2}, which preserves K's stratedy. □

References:

$http://arxiv.org/pdf/1109.4103v1.pdf$

CCS Discrete Math Lecture 4

$http://math.ucsb.edu/\ padraic/ucsb_2014_15/ccs_discrete_w2015/ccs_discrete_w2015_lecture4.pdf$