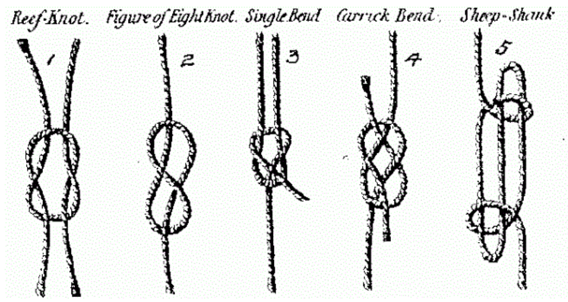
CCS Problem-Solving II | Professor: Padraic Bartlett

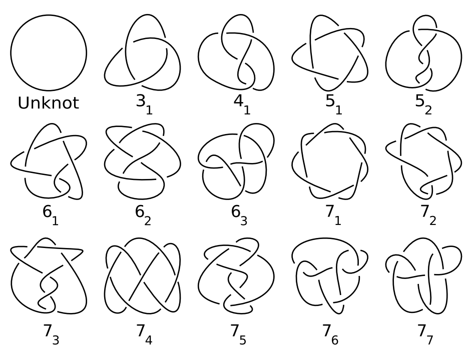
Project 6: Knot Theory

Weeks 1-10 | *UCSB 2015*

Outside of mathematics, **knots** are ways to loop a single piece of string around itself:

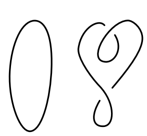


In mathematics, we mean something slightly different by a knot. Instead of a way to tie a rope, we will define a **knot** to be any way to embed[1] a circle in $\mathbb{R}^3$. Even though knots are three-dimensional things, we can draw them on paper:



Notice how we convey all of the information about what parts of the knot go "above" or "below" other parts by creating a break in the strand that goes under.

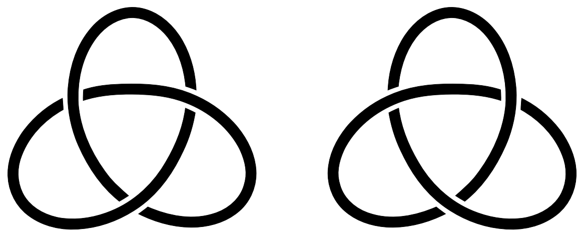
We consider two knots to be **equivalent** if there is some way to bend[2] space so that one knot is taken to the other. This is best illustrated by an example: in the picture below, both of these knots are the "same," because we can turn the first into the second by stretching/moving it around in space!



[1] An **embedding** of a circle in $\mathbb{R}^3$ is any map $\varphi : [0,1] to \mathbb{R}^3$ that is differentiable and injective, such that $\varphi(0) = \varphi(1)$. Think of it as any way of smoothly drawing a line in $\mathbb{R}^3$ without lifting your three-dimensional pen, so that you end back at where you started.

[2] Formally speaking, by "bend" we mean any deformation of $\mathbb{R}^3$. By *deformation*, we mean any map $\varphi : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ that is differentiable and bijective. Again, this is better thought of as taking space and moving/shifting/stretching it however you want, as long as you don't tear it or punch holes in it. Talk to me to see why the formal definition above corresponds to this idea!

It would seem like many knots are different: for example, the "unknot," which is just a simple loop, should be different from the trefoil (labeled 3_1 above.) But how can we prove this? How can we distinguish other kinds of knots?



The "left" and "right" trefoils. Can you deform space in such a way that one of these knots is transformed into the other?

In this project, students will discover and learn how to distinguish many different kinds of knots. Depending on how the project goes, it is entirely likely that open problems in knot theory can be studied.