Crossing Game Strategies

Chloe Avery, Xiaoyu Qiao, Talon Stark, Jerry Luo

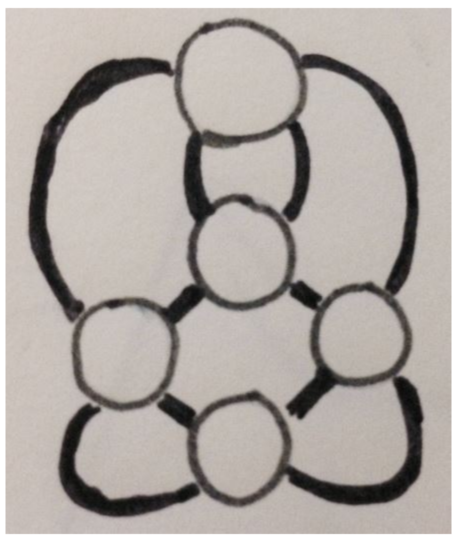
March 5, 2015

1 Strategies for Specific Knots

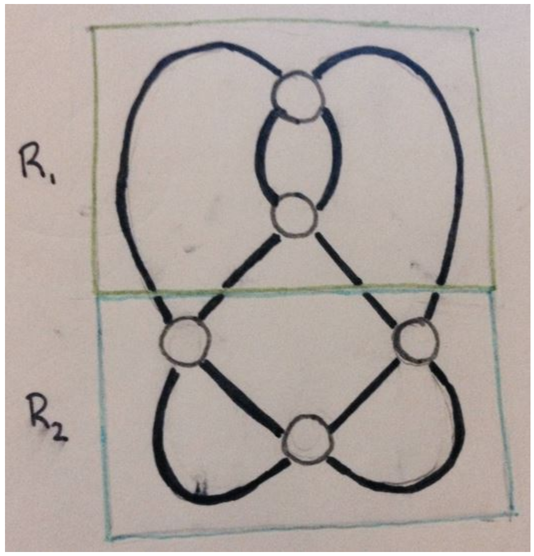
The following are a couple of crossing game boards for which we have found which player has a winning strategy.

1.1 Strategy for 5_2

The game board for the crossing game played on the 5_2 looks like the following:



We will first break this game into two regions, R_1 and R_2

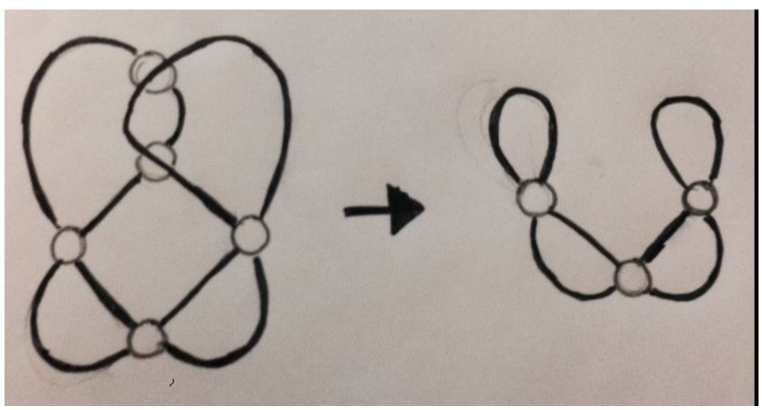


We will break the game into two cases:

- Player K goes first

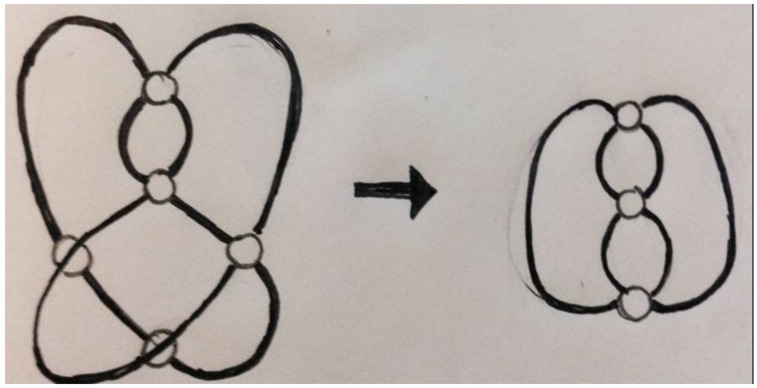
  Player K can move in one of:

  a. R_1
     If Player K moves in R_1, then Player U can make the same move as Player K in this region, thereby making two consecutive overs in R_1. If this happens, we have the following:



  Which we can see will be the unknot regardless of the moves made. Therefore, if Player K moves in R_1, Player U will win.

  b. R_2
     If Player K moves in R_2, then Player U can make the same move as Player K in this region, thereby making two consecutive overs in R_2. We can see that this reduces in the following way:



Which is simply another projection of the trefoil. Recall that Player U wins on the trefoil regardless of who goes first. Therefore, if Player K moves in R_2, Player U wins.

Therefore, if Player K goes first, Player U has a winning strategy.

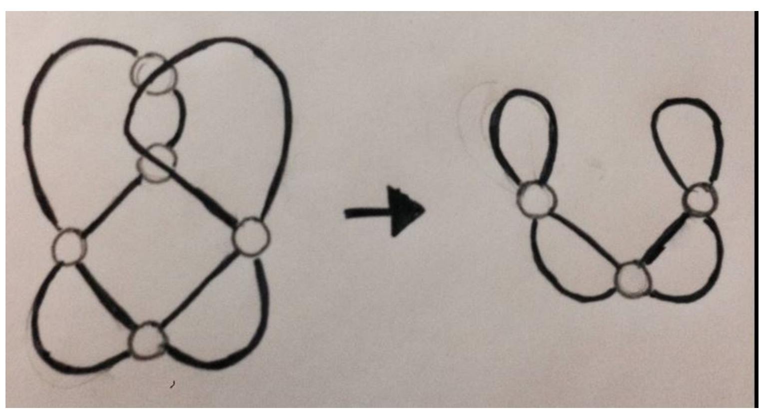
- Player U goes first

  To play optimally, Player U should make the first move in R_2 (this allows Player U to counter all moves that Player K makes, or rather it allows Player U to make consecutive overs with any move that Player K makes later).

  Player K makes the second move. Player K can move in one of:

  a. R_1

  If Player K moves in R_1, then Player U can make the same move as Player K in this region, thereby making two consecutive overs in R_1. If this happens, we have the following:

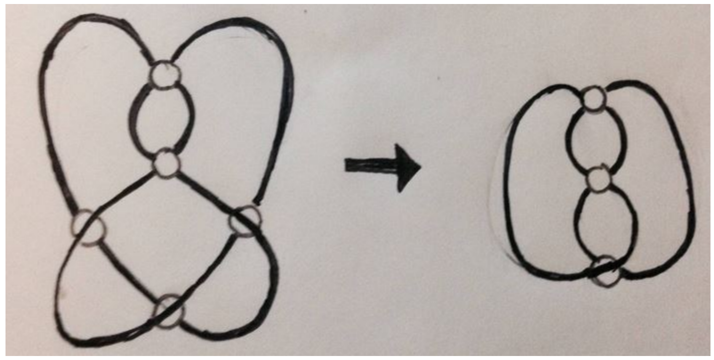


  except one of the moves in r_2 will be filled in, which means that in the decomposed version, one of the three moves will be filled in.

  We can see that this will be the unknot regardless of the moves made. Therefore, if Player K moves in R_1, Player U will win.

  b. R_2

  If Player K moves in R_2, then Player U can make the same move as Player K in this region, thereby making two consecutive overs in R_2. We can see that this reduces in the following way:



Which is simply another projection of the trefoil with one of the moves filled in. Recall that Player U wins on the trefoil regardless of who goes first. Therefore, if Player K moves in R_2, Player U wins.

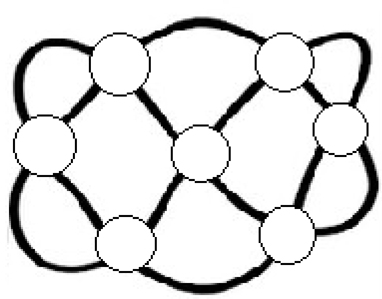
Therefore, if Player U goes first, Player U has a winning strategy.

Therefore, regardless of who goes first, Player U has a winning strategy.

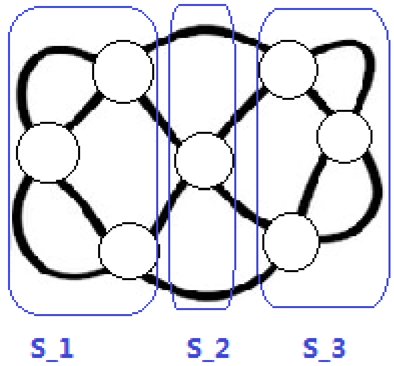
1.2 Strategy for Knot 7_4

In mathematical Knot Theory, 7_4 is the name of a 7-crossing knot which can be visually depicted in a highly-symmetric form, and so appears in the symbolism and/or artistic ornamentation of various cultures.

Now, consider the following diagram below. If we play the Crossing Game on it, which player has a winning strategy?



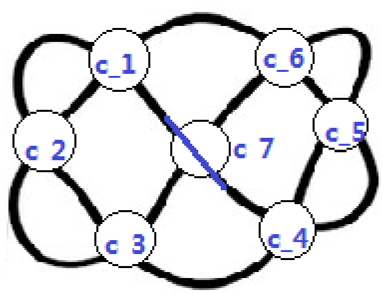
We can consider the strategies case by case.
Divide the knot diagram into three regions. Call them S_1, S_2, S_3. It is obvious S_1 and S_3 are symmetric.



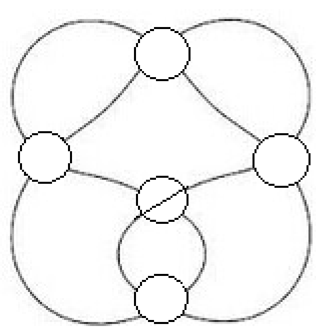
- Player U goes first

  Proposition 1. *If Player U goes first in S_2, then Player U can win.*

  Proof. If Player U takes the following step (it is actually the same case as making the opposite choice), then we notice that the unspecified crossing c_1 is symmetric with c_4, c_2 is symmetric with c_5, and c_3 is symmetric with c_6.

  Notice that if we can make either pair of $\{c_1, c_2\}$, $\{c_3, c_2\}$, $\{c_4, c_5\}$, or $\{c_5, c_6\}$ be both under-crossing or both over-crossing, this diagram can be changed into a 5_2.
  For example, when $\{c_1, c_2\}$ are both over-crossing or under-crossing, we have the graph below.

  We've proved that for knot 5_2, no matters who goes first, player U can always win. Therefore, we can consider this diagram as the case in which K has made his first step and its U's turn. Then U can win in the end. □

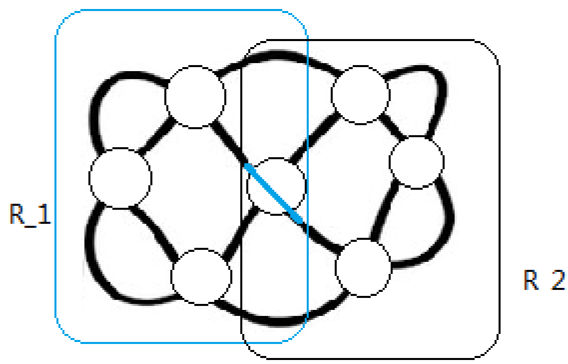
  Thus Player U has a winning strategy if Player U goes first. Now, let us consider the other case.

- Player K goes first

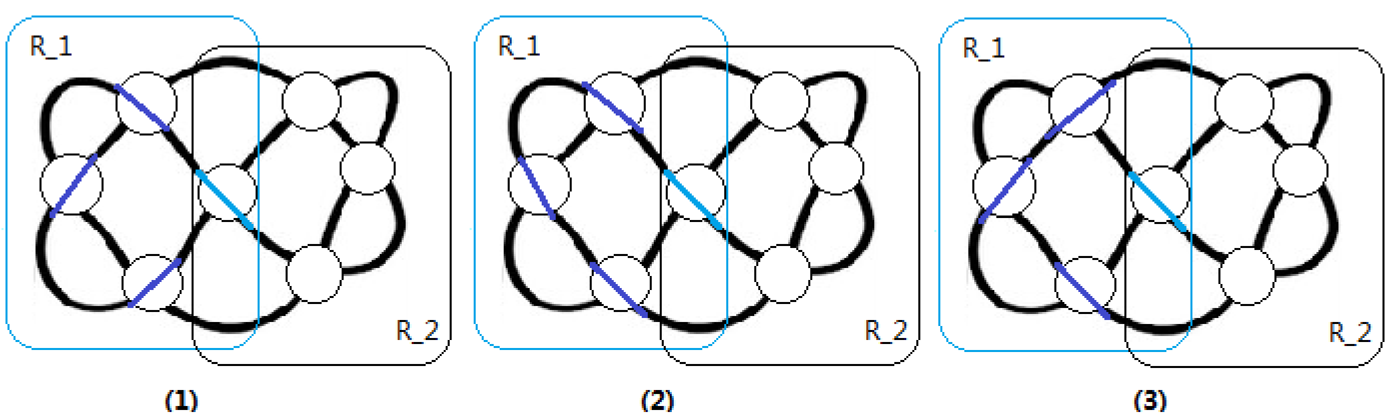
  Proposition 2. *If Player K goes first, then Player K can win.*
  Lemma 1. *If Player K makes the first move in S_1 or S_3, then Player U can win.*

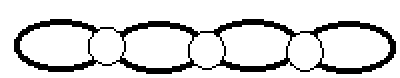
  Proof. Similar to our first proof, if we can make either pair of $\{c_1, c_2\}$, $\{c_3, c_2\}$, $\{c_4, c_5\}$, or $\{c_5, c_6\}$ be both under-crossing or both over-crossing, this diagram can be changed into a 5_2, which guarantees that Player U can win. □

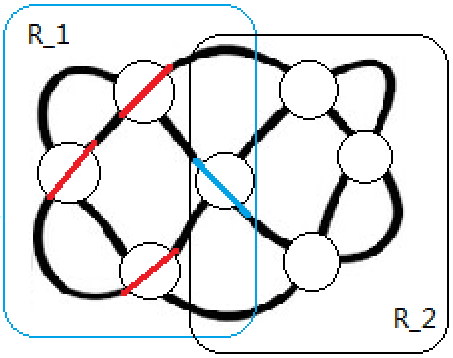
  Therefore, if Player K wants to win, Player K can make the first move in neither S_1 nor S_3. **What about if Player K makes its first move in S_2?** We can consider the region R_1 and R_2 respectively.



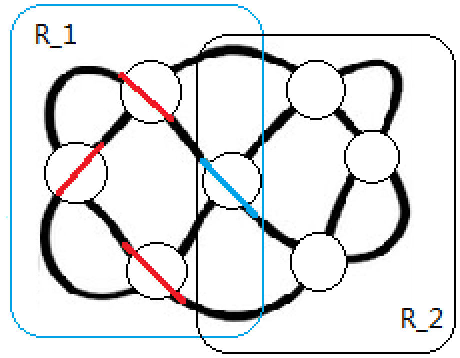
We call Region R_1 **"unlocked"** if it is in the following figuration.



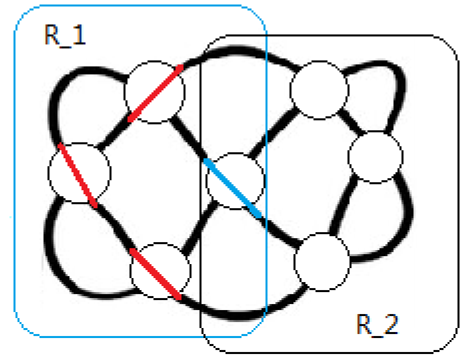
The case that R_2 being "unlocked" is equivalent to the case of R_1. This is easily seen by rotating the graph 180 degrees. Thus if the diagram reaches either of the above states, we can split two ends of the knot apart. When simplified, we get the diagram 
The region R_1 is **locked** if it is in the following figuration.



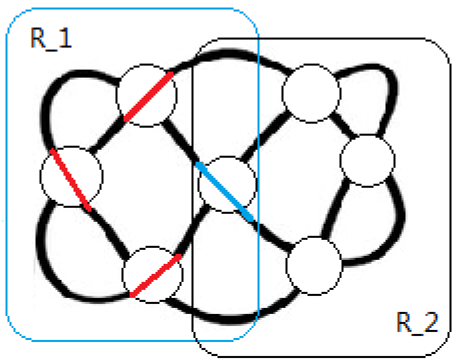
(4)



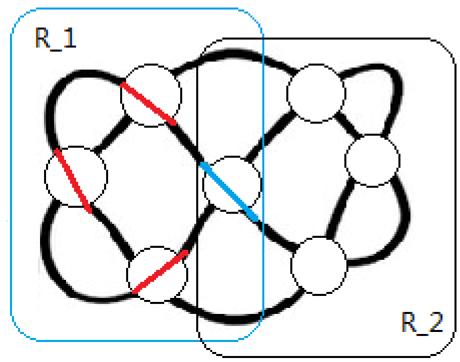
(5)



(6)

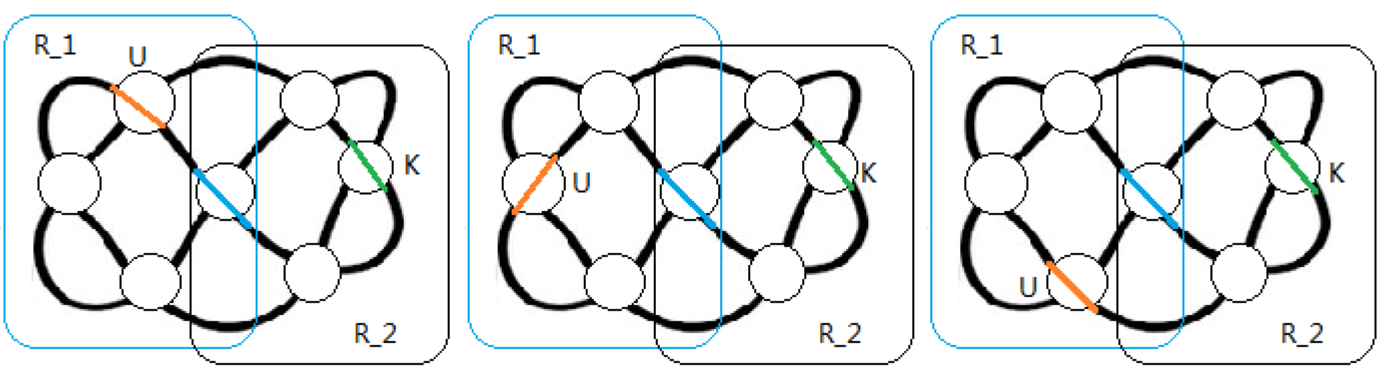


(7)



(8)

We can observe that **if both R_1 and R_2 are locked, then Player K wins.** Notice that if Player U makes the second move optimally and Player K makes the third move in the same region where Player U makes the second move, then Player U can win.
So we consider the case in which Player K makes the third move in a different region from the second move.
By observation, we find that to play optimally, U makes the second move in R_1 optimally, and after that, K makes the third move in R_2 and specifically in c_5, then K can win by **"locking"** both R_1 and R_2 .

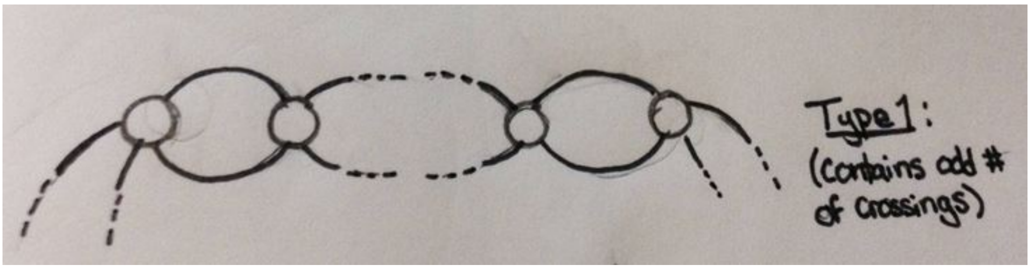


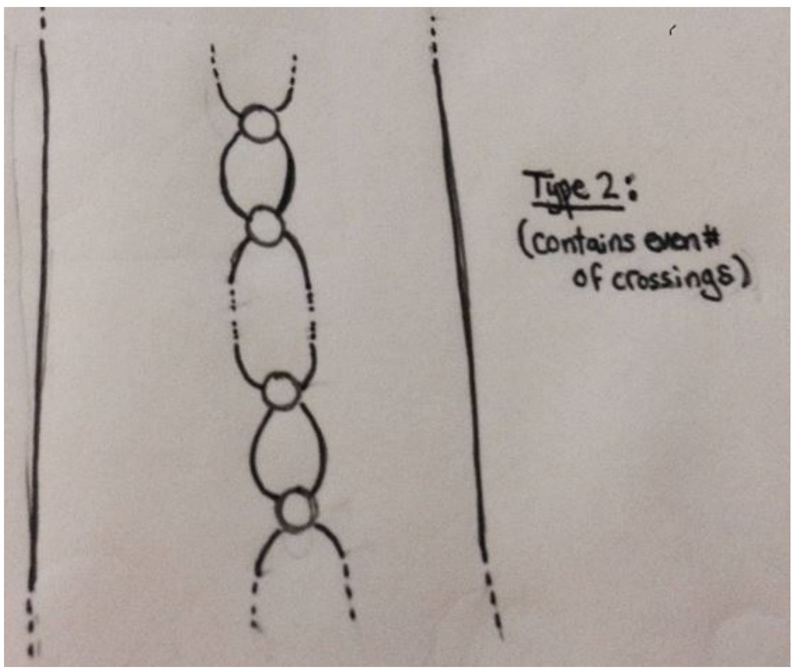
Now look at the last four unspecified crossings. It is Player U's turn. No matter what move Player U makes, Player K cab always counter the move that Player U made and **lock** both regions.

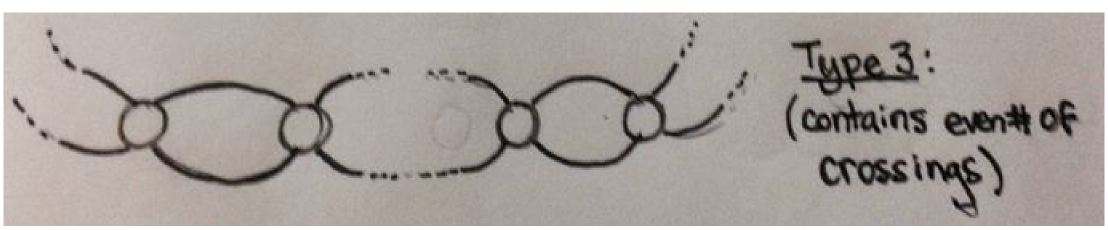
Therefore, the crossing game on 7_4 is a **P-Position** game, which means the first player can win.

2 Strategy for Generalized Odd Tri-Braid Knot

Definition 1. *An **Odd Tri-Braid Knot** is a knot that is constructed in the following way:*
We have 3 possible braid-like "pieces" that we can connect to make a knot of this type:





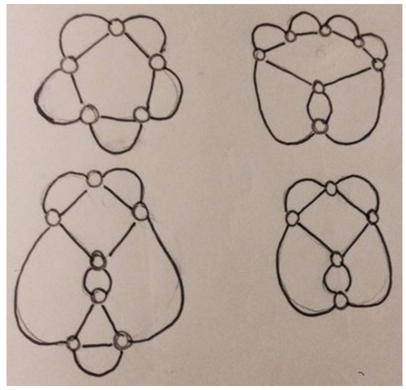


To construct an Odd Tri-Braid Knot, we connect the pieces according to the following rules:
(note that connections are made without adding any new crossings)
From top to bottom, we can have:

- *A piece of Type 1, then a piece of Type 2.*
- *A piece of Type 1 then a piece of Type 3.*
- *A piece of Type 1, then a piece of Type 2, then a piece of Type 3.*

2.1 Examples

The following are examples of Odd Tri-Braid Knots:



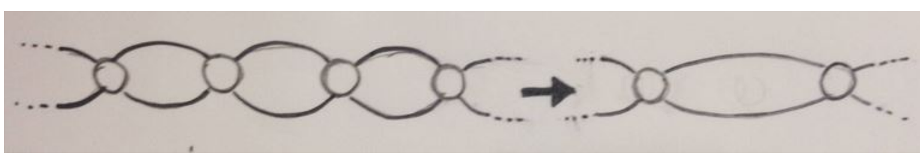
2.2 Crossing Game on Odd Tri-Braid Knots

I claim that Player U has a winning strategy for the crossing game played on an Odd Tri-Braid Knot

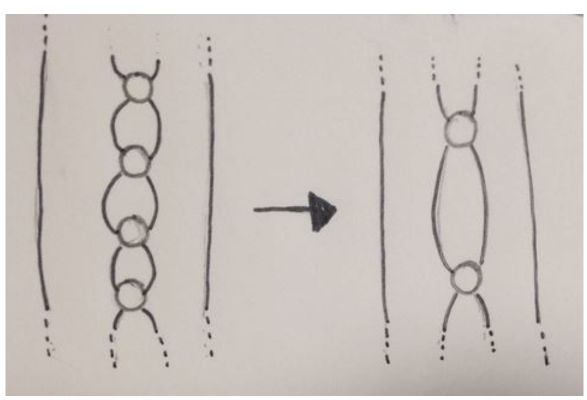
Note: We label the regions containing pieces of each separate type from top to bottom R_1, R_2, and R_3, respectively. If there are only two pieces, we label the regions containing pieces of each separate type from top to bottom R_1 and R_2, respectively.

Let us first talk about reductions on pieces of each type.

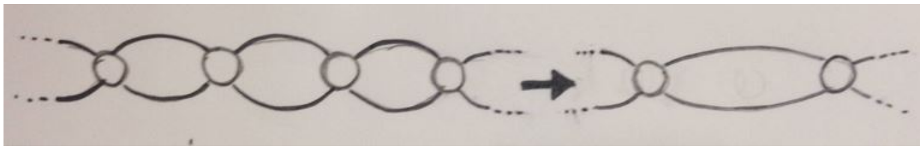
If two consecutive overs are made in a piece of Type 1 with n crossings, where n is odd, we can see that this reduces to a piece of Type 1 with $n-2$ crossings.



If two consecutive overs are made in a piece of Type 2 with n crossings, where n is even, we can see that this reduces to a piece of Type 2 with $n-2$ crossings.



If two consecutive overs are made in a piece of Type 3 with n crossings, where n is even, we can see that this reduces to a piece of Type 1 with $n-2$ crossings.



We will break the game into two cases:

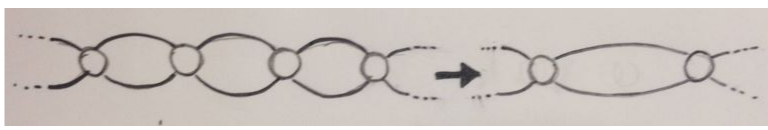
- Player U makes the first move.

  To play optimally, Player U should make the first move in R_1 (this allows Player U to counter all moves that Player K makes, or rather it allows Player U to make consecutive overs with any move that Player K makes later).

- Player K makes the first move, or any other time that Player K moves during the game.

  Player K can move in one of:

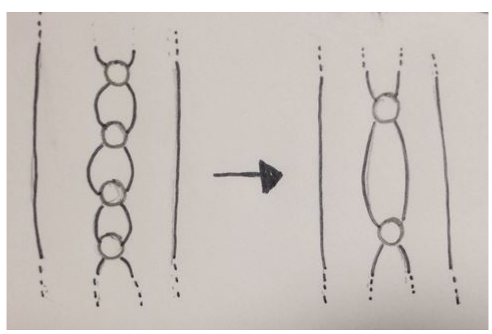
  a. If Player K moves in R_1, then Player U can make a move that gives us two consecutive overs in R_1. If this happens, we have a reduction in R_1, which is of Type 1.

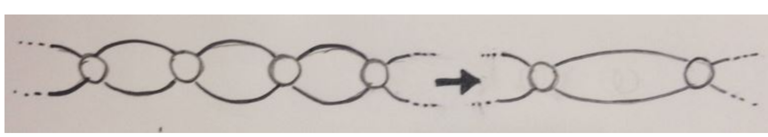
  

  Therefore, if Player K moves in R_1, Player U is able to make a reduction.

  b. R_2 or R_3

  If Player K moves in R_2 or R_3, then Player U can make the same move as Player K in this region, thereby making two consecutive overs in R_2 or R_3. If this happens, we have a reduction in either R_2 or R_3. Therefore, we have a reduction of a piece of either Type 2 or Type 3.

  Therefore, if Player K moves in R_2 or R_3, Player U is able to make a reduction.

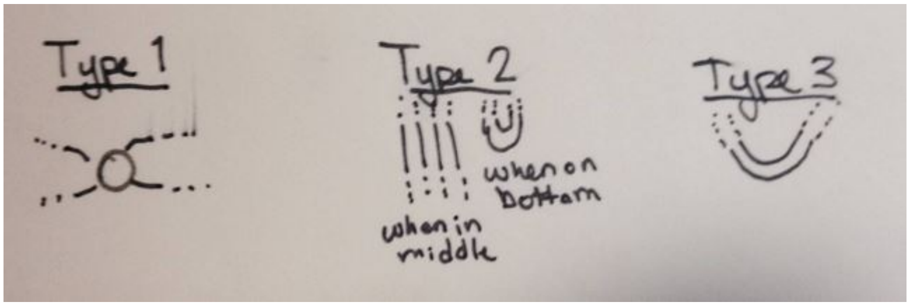
We notice that no matter the region in which Player K moves, Player U is able to make a reduction. So what happens if reductions continue to be made?

Note: Pieces of Type 1 cannot reduce completely as it contains an odd number of crossings, however, pieces of Type 2 or 3 can be reduced completely.

There are an odd number of pairs. We notice that reduction of our knot occurs on every pair of moves, with the exception of the first move if Player U goes first or with the exception of the last move if Player K goes first.

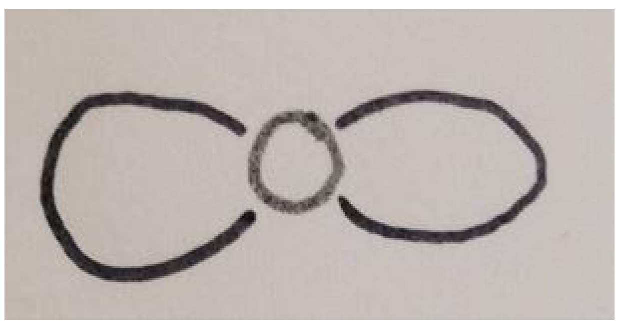
Therefore, we can look at the games after all possible reductions have occurred.

We first examine what pieces of each type will look like when reduced completely. We can see that applying all possible reductions takes pieces of Type 1, 2, or 3 to the following:

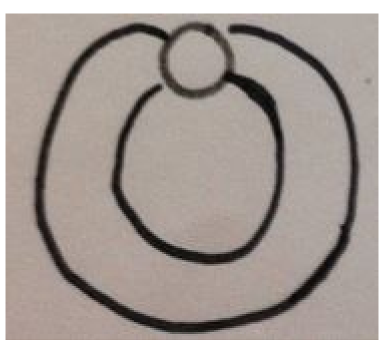


Therefore, we can examine the possibilities for our game after all possible reductions are made. Let an open circle in this case denote the one move that does not have a pair for reduction. Note that this move can be the first move if Player U goes first. Recall that we have 3 cases for our Odd Tri-Braid Knot. We can connect the reduced pieces of each type according to how we first constructed the knot. We have the following:

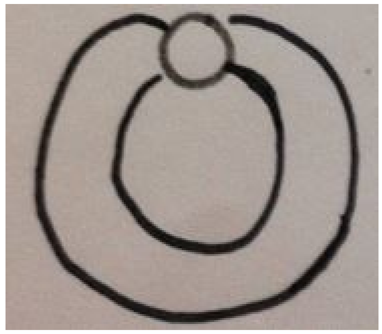
- A piece of Type 1 (R_1), then a piece of Type 2 (R_2) reduces to:



- A piece of Type 1 (R_1) then a piece of Type 3 (R_2).



- A piece of Type 1 (R_1), then a piece of Type 2 (R_2), then a piece of Type 3 (R_3).



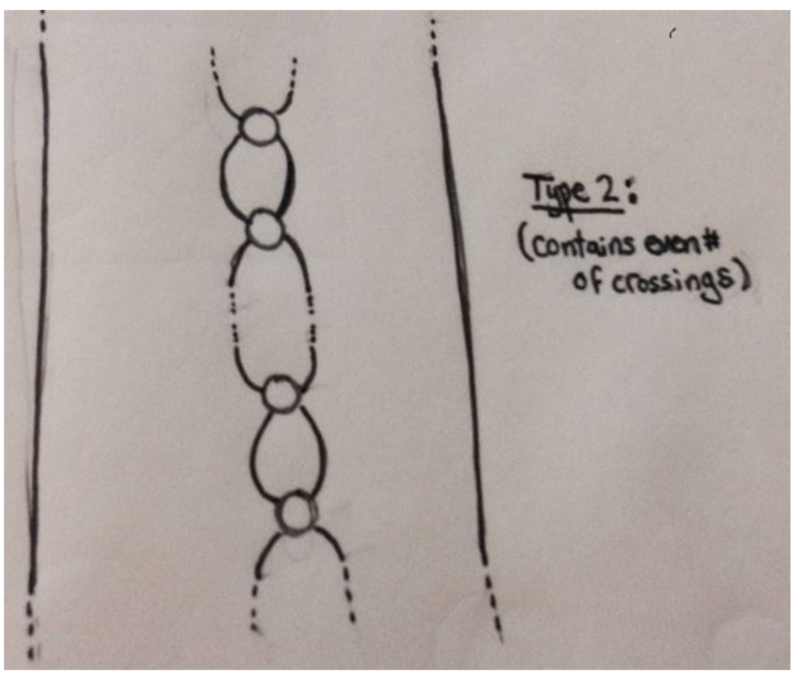
We notice that regardless of the move that is made in these reduced diagrams, they will all be the unknot. Therefore, we know that regardless of who goes first on an Odd Tri-Braid Knot, Player U has a winning strategy.

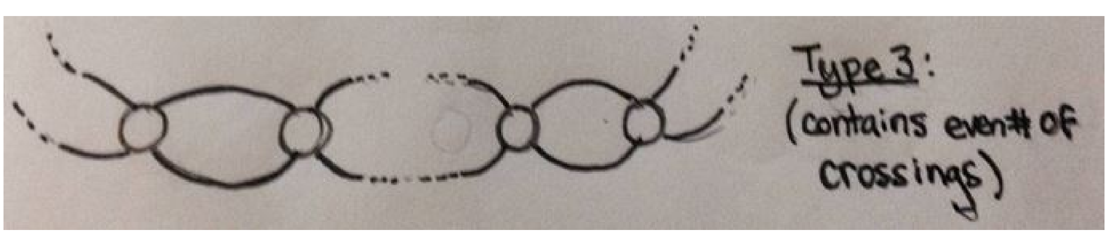
3 Crossing Game on Even Tri-Braid Knot

We have been trying to prove a strategy for knots with even number of crossings that are analogous to the Odd Tri-Braid Knot.

3.1 Structure of Even Tri-Braid Knot

Definition 2. *An* ***Even Tri-Braid Knot*** *is a knot that is constructed in the following way:*
We have 2 possible braid-like "pieces" that we can connect to make a knot of this type:





To construct an Even Tri-Braid Knot, we connect the pieces according to the following rules: (note that connections are made without adding any new crossings)
From top to bottom, we can have:

- *A piece of Type 3, then a piece of Type 2.*
- *A piece of Type 3, then a piece of Type 2, then a piece of Type 3.*

3.2 Problems With Even Tri-Braid Knot

For the Odd Tri-Braid Knot, we showed that no matter what Player K did, Player U could counter the move and make a reduction. This is also the case for the crossing game played on an Even Tri-Braid Knot when Player K goes first. Player U was able to do this when Player U goes first as well by making the extra move in the piece containing an odd number of crossings. However, this cannot happen in the Even Tri-Braid Knot. This allows Player K to counter moves that Player U makes.

We are trying to come up with a technique for dealing with this. We might consider thinking about each piece as a separate game and looking at what happens when separate crossing games are "summed". This leads us to study the game Nim, which we introduce below.

4 Nim

4.1 Impartial Combinatorial Games

General combinatorial are games that satisfy the following properties:

- There are two players.
- There is a finite set of positions available in the game (only on rare occasions will we mention games with infinite sets of positions).
- Rules specify which game positions each player can move to.
- Players alternate moving.
- The game ends when a player can't make a move.
- The game eventually ends in a finite amount of moves.

In this lecture we'll talk specifically about impartial games.

An **Impartial Combinatorial Game** is a combinatorial game in which the allowable moves depend only on the position and not on which of the two players is currently moving, and where the payoffs are symmetric (players are working towards the same objective). In other words, the only difference between player 1 and player 2 is that player 1 goes first.

Examples of impartial combinatorial games are Nim, sprouts, and green hackenbush while some other familiar games such as GO and chess are not considered impartial because the players do not access the same set of moves (they are called partisan).

4.2 Introduction to Nim

We will now look at the simple game of Nim, one of the most famous impartial combinatorial games which has led to some of the biggest advances in the field of combinatorial game theory. There are many versions of this game, but we will look at one of the most common.
Our motivation for studying the game of nim is because of the fact that all impartial combinatorial games may be reduced to games of Nim!
The game of Nim is played as follows. There are three piles of chips containing x_1, x_2, and x_3 chips respectively. (It is trivial to see that $x_1, x_2, x_3 \in \mathbb{N}$. We denote a specific initial setup of a game as x_1, x_2, x_3). Two players take turns moving. Each move consists of selecting one of the piles and removing chips from it. You may not remove chips from more than one pile in a single turn, but from the pile you selected you may remove as many chips as desired, from one chip to the whole pile. It is important to not that you may not skip your turn. The winner is the player who removes the last chip.

4.3 Basic Nim Analysis

In the game of Nim, there is exactly one terminal position (a position to where if someone arrives to it, they win), namely $(0, 0, 0)$, which is therefore a P-position. The solution to a one-pile game of Nim is trivial: the first player simply removes the whole pile. Any position with exactly one non-empty pile, say $(0, 0, x)$ with $x > 0$ is therefore an N-position. Consider a two-pile game of Nim. It is easy to see that the P-positions are those for which the two piles have an equal number of chips ($(0, 1, 1)$, $(0, 2, 2)$, etc). This is because if it is the opponent's turn to move from such a position, he must change to a position in which the two piles have an unequal number of chips, and then you can immediately return to a position with an equal number of chips (perhaps the terminal position). If all three piles are non-empty, the situation is more complicated. Clearly, $(1, 1, 1)$, $(1, 1, 2)$, $(1, 1, 3)$ and $(1, 2, 2)$ are all N-positions because they can be moved to $(1, 1, 0)$ or $(0, 2, 2)$. The next simplest position is $(1, 2, 3)$ and it must be a P-position because it can only be moved to one of the previously discovered N-positions. We may go on and discover that the next most simple P-positions are $(1, 4, 5)$, and $(2, 4, 6)$, but it is difficult to see how to generalize this. In order to further analyze the game strategies and positions in the game of Nim, we will have to utilize a concept known as the Nim sum.

4.4 Nim Sums

A vital concept to in the analysis of the game of Nim is called **Nim Sum**.

The **Nim Sum** ($\oplus$) of two non-negative integers is their sum without "carrying" in base 2. Let us make this notion precise. Every non-negative integer x has a unique base 2 representation of the form $x = x_m 2^m + x_{m1} 2^m 1 + \ + x_1 2 + x_0$ for some $m \in \mathbb{N}$, where each x_i is either 0 or 1. We use the notation $(x_m x_{m1} \cdots x_1 x_0)_2$ to denote this representation of x to the base two. For example, $22 = 1 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1 = (10110)_2$. The nim-sum of two integers is found by expressing the integers to base two and using addition modulo 2 on the corresponding individual components:

Definition 3. *The nim sum of $(x_m \cdots x_0)_2$ and $(y_m \cdots y_0)_2$ is $(z_m \cdots z_0)_2$, and we write $(x_m \cdots x_0)_2 \oplus (y_m \cdots y_0)_2 = (z_m \cdots z_0)_2$, where for all k, $z_k = x_k + y_k \mod 2$, that is, $z_k = 1$ if $x_k + y_k = 1$ and $z_k = 0$ otherwise.*

For example, $(10110)_2 \oplus (110011)_2 = (100101)_2$. This says that $22 \oplus 51 = 37$. This is easier to see if the numbers are written vertically (we also omit the parentheses for clarity):

$$
\begin{array}{r}
22 = \;\; 10110_2 \\
\underline{\oplus \;\; 51 = 110011_2} \\
\text{nim sum} = 100101_2 = 37
\end{array}
$$

Nim sum is associative (i.e. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$) and commutative (i.e. $x \oplus y = y \oplus x$), because of inheritence from addition in modulo 2. Thus, we may write $x \oplus y \oplus z$ without specifying the order of addition. Furthermore, 0 is an identity for addition ($0 \oplus x = x$), and every number is its own negative ($x \oplus x = 0$), so that the cancellation law holds: $x \oplus x \oplus y = x \oplus x \oplus z \implies y = z$. (If $x \oplus y = x \oplus z$, then $x \oplus x \oplus y = x \oplus x \oplus z$, and so $y = z$.)

One of the reasons that the Nim Sum is using for analyzing nim is because of the following theorem.

Theorem 1. *A position, $(x_1, x_2, x_3, \cdots, x_n)$, in n-piled Nim (a more general variation) is a P-position if and only if the nim-sum of its components is zero, which means $x_1 \oplus x_2 \oplus x_3 \oplus \cdots \oplus x_n = 0$.*

A proof for this theorem will be omitted for the conservation of space.

4.5 Relating to the Crossing Game

Recall that we needed a strategy for finding out who wins on the crossing game (for Even Tri-Braid Knots specifically). We were considering breaking our knots into separate games, thereby giving us sums of games. A game where separate games sum nicely is Nim. We are hoping to gain some insight into how to sum Crossing games by looking at Nim. In the game of Nim, we may "add up" multiple games together by taking the Nim sum of all the piles. This tells us whether the resulting game is a P or an N position game. In a similar manner, we find what position a certain knot is by breaking it into regions and analyzing the individual regions. This relation comes with a variation however, due to the fact that the game of Nim is impartial with each player having the same objective. In the crossing game, on the other hand, each player is working toward a different objective by manipulating the knot with each move in attempt for it to become it respective "type" (unknot or knot).
Using this tool, we will hopefully be able to further analyze the Even Tri-Braid Knots (and others) in an attempt to determine their position.

5 References

1. **Winning Ways for your Mathematical Plays** by: Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy
2. $http://www.math.washington.edu/\ morrow/papers/will-thesis.pdf$
3. $http://faculty.smcm.edu/sganzell/papers/untangle-2.pdf$
4. $http://www.math.ucla.edu/\sim tom/Game\_Theory/comb.pdf$
5. $http://web.mit.edu/sp.268/www/nim.pdf$