Lecture 4: Integrals in $\mathbb{R}^n$

This is the third week of the Mathematics Subject Test GRE prep course; here, we review the concepts of **integrals** in higher dimensions!

1 Definitions and Concepts

We start by reviewing the definitions/theorems we have for the integrals of functions on $\mathbb{R}^n$:

1. **Types of integrals.** You've (in theory) learned how to take several kinds of integrals in undergrad:

   - **"Normal" integrals.** Given a region $R \subset \mathbb{R}^n$, we know how to take the integral of any function $F : \mathbb{R}^n \to \mathbb{R}^m$ over such a region by taking **iterated integrals**. For example, if R is some sort of a n-dimensional box $[a_1, b_1] \times \ldots [a_n, b_n]$, we can write $\iint_R F dV$ as the iterated integral

     $$\int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} F dx_n \ldots dx_1.$$

     Part of being able to do these integrals is the ability to describe a region R via sets of nested parameters. For example, if R is the upper-right quadrant of the unit disk

     $$R = \{(x, y) : x^2 + y^2 \leq 1, 0 \leq x, 0 \leq y\},$$

     you should be able to describe R as the set of all points such that

     $$x \in [0, 1], y \in [0, \sqrt{1 - x^2}],$$

     and therefore notice that that we can express

     $$\iint_R f(x, y) dydx = \int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y) dydx,$$

     for some function f. Be able to do this "nested parameter" thing over most kinds of regions: usually, the way you do this is by picking one variable, determining its maximum range, then (for some fixed value of that first variable) pick a second variable and determine its maximum range depending on the first variable, and so on/so forth.

- **Line integrals.** Given a parametrized curve $\gamma : [a, b] \to \mathbb{R}^n$, we can find the integral of either a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ or a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ along this curve. Specifically, we can express these integrals as the following:

$$\int_\gamma F \cdot d\gamma = \int_a^b (F \circ \gamma(t)) \cdot (\gamma'(t))dt, \quad \text{and}$$
$$\int_\gamma f \; d\gamma = \int_a^b (f \circ \gamma(t))||\gamma'(t)||dt.$$

- **Surface integrals.** Given a parametrized surface S with parametrization $T : R \to S$, $R \subseteq \mathbb{R}^2$, we can find the integral of any function $f : \mathbb{R}^n \to \mathbb{R}$ over S, as well as the integral of any vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ over S. Specifically, we can express the integral of f over S as the following two-dimensional integral over R:

$$\iint_S f dS = \iint_R (f \circ T(u, v)) \cdot ||T_u \times T_v|| \, dudv.$$

  As well, recall that a unit normal vector to our surface, $\ltimes$, can be given by the formula

$$\mathbf{n} = \frac{(T_u \times T_v)}{||T_u \times T_v||} \text{ or } \frac{(T_v \times T_u)}{||T_v \times T_u||}$$

  up to the **orientation** of $\mathbf{n}$: i.e. depending on whether we look at $(T_u \times T_v)$ or $(T_v \times T_u)$, we will get either $\mathbf{n}$ or $-\mathbf{n}$. Choosing an **orientation** for our surface S is simply choosing which of these two choices of normal vectors we will make for our entire integral: whenever we ask you to integrate a vector field over a surface, we will tell you what orientation you should pick (i.e. by asking you to orient S so that "the normals point away from the origin," or something like that.) Once you've fixed an orientation, say the $T_u \times T_v$ one, we define the integral of F over S as the following integral:

$$\iint_S F \cdot dS = \iint_S F \cdot \mathbf{n} dS = \iint_R (f \circ T(u, v)) \cdot \frac{(T_u \times T_v)}{||T_u \times T_v||} \cdot ||T_u \times T_v|| \, dudv$$
$$= \iint_R (f \circ T(u, v)) \cdot (T_u \times T_v) \, dudv.$$

  The trickiest thing going on here is "how" you **choose** your parametrization. For finding a parametrization of a surface S, you can usually do one of the following two things:

– Often, if you describe your surface S in cylindrical or spherical coördinates, you'll see that one of the coördinates you're describing your surface in is constant. For example, a spherical shell of radius 3 can be described in spherical coördinates as the set of all point $(3, \theta, \phi)$, where $\theta \in [0, 2\pi], \phi \in [0, \pi]$. In this kind of situation, our parametrization is just using this coördinate system with the constant variable treated as a constant: i.e. for the spherical shell of radius 3, our parametrization is just

$$T(\theta, \phi) = (3\cos(\theta)\sin(\phi), 3\sin(\theta)\sin(\phi), 3\cos(\phi)),$$

where $\theta \in [0, 2\pi], \phi \in [0, \pi]$.

– If this doesn't work out, the other tactic that's often useful is finding an equation that describes your surface, and solving for one of the variables in terms of the others. For example, suppose that we're looking at the surface S given by the upper sheet of the hyperboloid of two sheets between heights 1 and 2: i.e.

$$S = \{(x, y, z) : -x^2 - y^2 + z^2 = 1, z \in [1, 2]\}.$$

In this case, because z is positive, we can solve for z in terms of the other variables, and express S as

$$S = \{(x, y, z) : z = \sqrt{1 + x^2 + y^2}, z \in [1, 2]\}.$$

We can then use this to formulate a parametrization of S: simply let x and y range over the possible values that keep z between 1 and 2, and then set $z = \sqrt{1 + x^2 + y^2}$:

$$T(x, y) = (x, y, \sqrt{1 + x^2 + y^2}), x \in [-\sqrt{3}, \sqrt{3}], y \in [-\sqrt{3 - x^2}, \sqrt{3 - x^2}].$$

You can of course combine these two approaches: for example, if we were to use cylindrical coördinates on our surface S above and replace x with $r\cos(\theta)$, y with $r\sin(\theta)$, we can see that we can easily express T instead as the map

$$T(r, \theta) = (r\cos(\theta), r\sin(\theta), \sqrt{1 + r^2}), r \in [0, \sqrt{3}], \theta \in [0, 2\pi],$$

which may be easier to work with.

2. **Tools for evaluating integrals.** Throughout your undergraduate career, you've ran into many integrals of the above kinds that were difficult or impossible to directly evaluate. Motivated by these problems, we developed a number of theorems and tools about integration, which we repeat here:

- **Green's theorem.** There are a number of forms of Green's theorem; we state the simpler and most commonly used version here. Suppose that R is a region in $\mathbb{R}^2$ with boundary ∂R given by the simple closed curve C, and suppose that γ

is a traversal of C in the counterclockwise direction. Suppose as well that P and Q are a pair of C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}$. Then, we have the following equality:

$$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\gamma (Pdx + Qdy)\,.$$

- **Stokes' theorem.** Stokes' theorem, quite literally, is Green's theorem for surfaces in $\mathbb{R}^3$ (as opposed to restricting them to lying in the plane $\mathbb{R}^2$.) Specifically, it is the following claim: suppose that S is a surface in $\mathbb{R}^3$ with boundary ∂S given by the simple closed curve C, suppose that $\mathbf{n}$ is a unit normal vector to S that gives S some sort of orientation, and suppose that γ is a traversal of C such that the interior of S always lies on the left of γ's forward direction, assuming that we're viewing the surface such that the normal vector $\mathbf{n}$ is pointing towards us. Suppose as well that F is a vector field from $\mathbb{R}^3$ to $\mathbb{R}^3$. Then, we have the following equality:

  $$\iint\limits_S (\nabla \times F) \cdot \mathbf{n}\ dS = \int_\gamma F d\gamma.$$

  In general, you use Green's and Stokes's theorems whenever you have a integral of a function over an awful curve (and taking derivatives to work with your function over a region, which is what the curl does, will make things easier), or you have an integral of a curl-like function over an awful region (and working on the curve would make things easier.)

- **Divergence/Gauss's theorem.** Let W be a region in $\mathbb{R}^3$ with boundary given by some surface S, let $\mathbf{n}$ be the outward-pointing (i.e. away from W) unit normal vector to S, and let F be a smooth vector field defined on W. Then

  $$\iiint\limits_W (\mathrm{div}(F))dV = \iint\limits_{\partial W} (F \cdot \mathbf{n})dS.$$

  Again, use this like you would use Green's and Stokes's theorems.

- **Change of variables.** A common tactic to make integrals easier is to apply the technique of change of variables, which allows us to describe regions in $\mathbb{R}^n$ using coördinate systems other than the standard Euclidean ones. In general, the change-of-variables theorem says the following:

  - Suppose that R is an open region in $\mathbb{R}^n$, g is a C^1 map $\mathbb{R}^n \to \mathbb{R}^n$ on an open neighborhood of R, and that f is a continuous function on an open neighborhood of the region $g(R)$. Then, we have

    $$\int_{g(R)} f(\mathbf{x})dV = \int_R f(g(\mathbf{x})) \cdot \det(D(g(\mathbf{x})))dV.$$

  Specifically, the three most common change-of-variable choices are transitions to the **polar**, **cylindrical**, and **spherical** coördinate systems, which we review here:

– **Polar coördinates.** Suppose that R is a region in $\mathbb{R}^2$ described in polar coördinates: i.e. there is some set $A \subseteq [0, \infty) \times [0, 2\pi)$ such that $\gamma(A) = R$, where γ is the polar coördinates map $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$. Then, for any integrable function $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\iiint_{\gamma(A)} f(x, y) dV = \iiint_{A} f(r\cos(\theta), r\sin(\theta)) \cdot r \; dV.$$

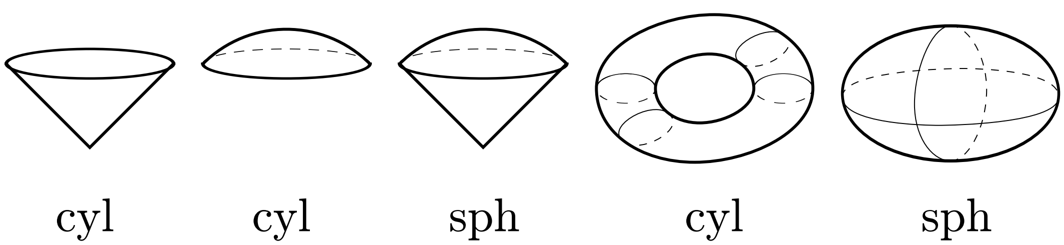
– **Cylindrical coördinates.** Suppose that R is a region in $\mathbb{R}^3$ described in cylindrical coördinates: i.e. there is some set $A \subseteq [0, \infty) \times [0, 2\pi) \times (-\infty, \infty)$ such that $\gamma(A) = R$, where γ is the cylindrical coördinates map $(r, \theta, z) \mapsto (r\cos(\theta), r\sin(\theta), z)$. Then, for any integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\iiint_{\gamma(A)} f(x, y) dV = \iiint_{A} f(r\cos(\theta), r\sin(\theta), z) \cdot r \; dV.$$

– **Spherical coördinates.** Suppose that R is a region in $\mathbb{R}^3$ described in spherical coördinates: i.e. there is some set $A \subseteq [0, \infty) \times [0, 2\pi) \times [0, \pi)$ such that $\gamma(A) = R$, where γ is the spherical coördinates map $(r, \theta, \varphi) \mapsto (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$. Then, for any integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\iiint_{\gamma(A)} f(x, y) dV = \iiint_{A} f(r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi)) \cdot r^2 \sin(\varphi) \; dV.$$

One of the trickiest things to do with change of variables is deciding which coördinate system to use on a given set. For example, consider the following five shapes:



To describe the cone, sphere cap, or torus above, cylindrical coördinates are probably going to lead to the easiest calculations. Why is this? Well, all three of these shapes have a large degree of symmetry around their z-axis; therefore, we'd expect it to be relatively easy to describe these shapes as a collection of points (r, θ, z). However, these shapes do *not* have a large degree of rotational symmetry: in other words, if we were to attempt to describe them with the coördinate (r, θ, φ), we really wouldn't know where to begin with the φ coördinate.

However, for the ellipsoid and "ice-cream-cone" section of the ellipsoid, spherical coördinates are much more natural: in these cases, it's fairly easy to describe these sets as collections of points of the form (r, θ, φ).

In general, if you're uncertain which of the two to try, simply pick one and see how the integral goes! If you chose wisely, it should work out; otherwise, you can always just go back and try the other coördinate system.

3. **Applications of the integral.** Finally, it bears noting that we've developed a few applications of the integral to finding volume, surface area, length, and centers of mass. We review these here:

   - **Volume, surface area, and length**. If you have a solid V, a surface S, or a curve C, you can find the volume/area/length of your object by integrating the function 1 over that object.
   - **Area, via Green's theorem.** If you have a region $R \subset \mathbb{R}^2$ with boundary given by the counterclockwise-oriented curve γ, you can use Green's theorem to find its area as a line integral. Specifically, notice that if $F(x, y) = \left(-\frac{y}{2}, \frac{x}{2}\right)$, we have $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, and therefore that Green's theorem says that

     $$\iint\limits_R 1 \, dA = \int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma.$$

   - **Center of mass.** Suppose that an object A (a curve, surface, or solid) has density function $\delta(\mathbf{x})$. Then, the x_i-coördinate of its center of mass is given by the ratio

     $$\frac{\int\limits_A x_i \delta(\mathbf{x}) \, dA}{\int\limits_A \delta(\mathbf{x}) \, dA}.$$

To illustrate these concepts, we work some examples:

2 Example Problems

Question 1. *Let S denote the cut-off paraboloid surface formed by the equations $z + 1 = x^2 + y^2$, $z \leq 0$, oriented so that the z-component at the origin is positive. Let F denote the vector field $F(x, y, z) = \left(e^z y, e^{z^2} x, e^{z^3} z\right)$. Find the integral of $\nabla \times F$ over S.*

Solution. First, we calculate $\nabla \times F$:

$$\begin{aligned}\nabla \times F &= \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right) \\ &= \left(0 - 2e^{z^2} xz, e^z y - 0, e^{z^2} - e^z\right).\end{aligned}$$

You could parametrize S and directly integrate this vector over S. But this looks **awful.** Instead, what we can do is use Stokes' theorem! In particular, consider the surface D given by the unit disk $x^2 + y^2 \leq 1, z = 0$. This surface has the same boundary as our surface S: specifically, $\partial S = \partial D = x^2 + y^2 = 1$. Suppose we orient the unit disk with the normal

$(0,0,1)$, which is normal to the unit disk everywhere. Then these boundaries have the same orientation, if both boundaries are oriented positively with respect to their corresponding surfaces.

Therefore, we can use Stokes's theorem once to see that

$$\iint\limits_S \nabla \times F \cdot dS = \int\limits_{\partial S^+} F \cdot ds,$$

and we can use it again to see that

$$\int\limits_{\partial D^+} F \cdot ds = \iint\limits_D \nabla \times F \cdot dS.$$

Because $\partial S = \partial D$, these integrals are all the same! So, to calculate $\iint\limits_S \nabla \times F \cdot dS$, we can calculate $\iint\limits_D \nabla \times F \cdot dS$. We do this here. Notice that the unit normal $\mathbf{n}$ to the unit disk as a surface in $\mathbb{R}^3$ is simply $(0,0,1)$; this saves us the effort of having to parametrize the disk, because

$$\iint\limits_D \nabla \times F \cdot dS = \iint\limits_D \left(0 - 2e^{z^2}xz, e^z y - 0, e^{z^2} - e^z\right)\Big|_{\text{any parametrization of the disk}} \cdot (0,0,1)\ dS = 0,$$

as any parametrization of the disk will have zero z-coördinate, and thus our integral is of the form $(-,-,0) \cdot (0,0,1) = 0$!

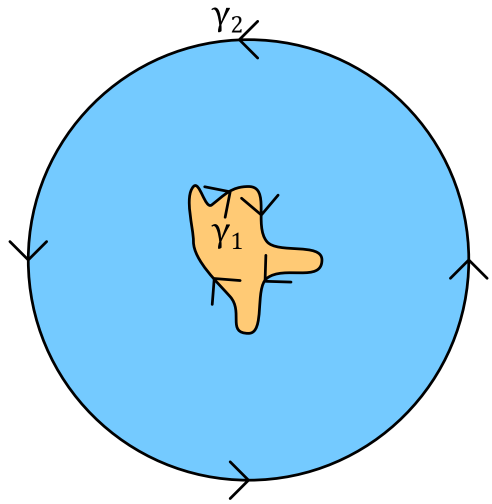
Lots of set-up, but it makes our calculations trivial: we didn't even have to parametrize the unit disk! This is one of the cooler applications of Stokes's theorem: switching between different surfaces.

You can also use things like Stokes's and Green's theorem to switch integrals between different curves: it's a little weirder, but sometimes is really useful.

Question 2. *Take a pond whose outer perimeter is given by a circle of radius 4 and contains 16π cubic centimeters of water. Drop a rock in the center of the pond. Assume that the rock's edges are roughly vertical, i.e. we can model the boundary of the rock in the pond as some 2-d shape. After doing this, assume the water has height h in centimeters.*

Suppose that there is an ant walking around the boundary of the rock. Suppose further that this ant is being blown on by a wind current, which imparts force on the ant corresponding to the vector field $\mathbf{F}(x,y) = (-y,x)$. In one walk of the ant around the boundary of the rock, how much energy does the wind impart on the ant? In other words, what is $\int_{\gamma_1} \mathbf{F} \cdot ds$?

Solution. We draw the situation here.



As labelled above, let γ_1 denote the perimeter of the rock, and γ_2 denote the perimeter of the pond. Let R denote the region between the outer curve and the inner curve. We want to calculate

$$\int_{\gamma_1} \mathbf{F} \cdot dS.$$

This is... hard, because, well, we don't actually know what γ_1 **is**. However, we can get around this with Green's theorem!

In particular: notice that Green's theorem says that the integral of $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ over R is equal to the integral of $\mathbf{F}$ over the two boundary components γ_1, γ_2, provided that they're both oriented (as drawn) so that R is always on the left-hand-side of each curve. In other words,

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \int_{\gamma_1} \mathbf{F} \cdot ds + \int_{\gamma_2} \mathbf{F} \cdot ds.$$

So, we can solve for the integral we want to study, in terms of two other integrals:

$$\int_{\gamma_1} \mathbf{F} \cdot ds = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA - \int_{\gamma_2} \mathbf{F} \cdot ds.$$

These are, surprisingly, things we can calculate. In specific, we have that

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R (1 - (-1))\, dA = \iint_R 2dA = 2 \cdot (\text{surface area of } R).$$

Because the pond started with 16π cubic centimeters of water and had height h after we dropped the rock in, we know that R has surface area $\frac{16\pi}{h}$, and therefore that this integral is $\frac{32\pi}{h}$.

As well, we can find $\int_{\gamma_2} \mathbf{F} \cdot ds$. We parametrize γ_2 as $\gamma_2(t) = (4\cos(t), 4\sin(t))$:

$$\int_{\gamma_2} \mathbf{F} \cdot ds = \int_0^{2\pi} (-4\sin(t), 4\cos(t)) \cdot (-4\sin(t), 4\cos(t))\, dt = \int_0^{2\pi} 16 dt = 32\pi.$$

Therefore, we can combine these two integrals to calculate $\int_{\gamma_1} \mathbf{F} \cdot ds$:

$$\int_{\gamma_1} \mathbf{F} \cdot ds = 32\pi \left(\frac{1}{h} - 1\right).$$

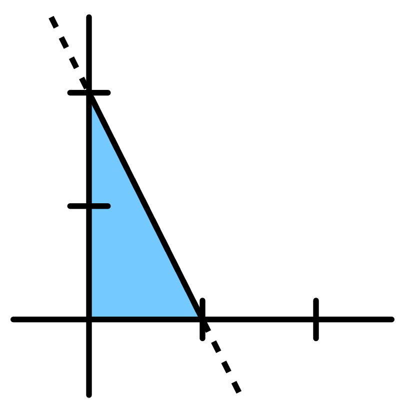
This is pretty cool: we know exactly how much work was done by this wind current, even though we have no idea what path we integrated over!

Question 3. *Let T be a triangle with vertices $(1,0,0), (0,2,0), (0,0,3)$. If this triangle is made out of some material with uniform density across its surface, what is the x-coördinate of the center of mass of T?*

Solution. We want to find the x-coördinate of the center of mass of T. This is the "average" x-coördinate over our entire surface. Recall the following: if we want to find the average value of a function f on a surface T, we want to find the integrals $\iint_T f dA$ and $\iint_T 1 dA$, and divide the first of these two integrals by the second: this gives us the average value of f over T.

So. We start by parametrizing our triangle. We do this by considering coördinates one-by-one. We first look at x: over our entire triangle, x ranges from 0 to 1.

We now look at the possible range of y-values, given x. We do this by projecting our triangle onto the xy-plane: there, this is the triangle with vertices $(0,0), (1,0), (0,2)$.



Given any fixed value of x, we can see that y ranges from 0 to $2 - 2x$.

Finally, we need to solve for z given x and y. To do this, we just need to find the plane this triangle lies in: this will give us an equation relating x, y and z. We do this by taking the generic equation for a plane

$$ax + by + cz = d$$

and plugging in the three points $(1,0,0), (0,2,0), (0,0,3)$ into this equation:

$$a = d, b = \frac{d}{2}, c = \frac{d}{3}.$$

This gives us that our plane has the equation

$$dx + \frac{d}{2}y + \frac{d}{3}z = d,$$

which (if we divide by d) becomes

$$x + \frac{y}{2} + \frac{z}{3} = 1.$$

Solving for z gives us

$$z = 3 - 3x - \frac{3y}{2}.$$

So, we can parametrize our triangle via the map $T(x, y) = \left(x, y, 3 - 3x - \frac{3y}{2}\right)$, where x ranges from 0 to 1 and (given x) y ranges from 0 to $2 - 2x$.

So, if we want to find $\iint_T 1dA$, we can just use this parametrization:

$$\begin{aligned}
\iint_T 1\ dA &= \int_0^1 \int_0^{2-2x} \left|\left|\frac{\partial T}{\partial x} \times \frac{\partial T}{\partial y}\right|\right| dxdy \\
&= \int_0^1 \int_0^{2-2x} \left|\left|\left(1, 0, -3\right) \times \left(0, 1, -\frac{3}{2}\right)\right|\right| dxdy \\
&= \int_0^1 \int_0^{2-2x} \left|\left|\left(3, \frac{3}{2}, 1\right)\right|\right| dxdy \\
&= \int_0^1 \int_0^{2-2x} \sqrt{9 + \frac{9}{4} + 1}\ dxdy \\
&= \int_0^1 \int_0^{2-2x} \frac{7}{2}\ dxdy \\
&= \int_0^1 7 - 7x\ dx \\
&= \frac{7}{2}.
\end{aligned}$$

Similarly, if we want to find $\iint_T xdA$, we can do mostly the same thing:

$$\begin{aligned}
\iint_T x\ dA &= \int_0^1 \int_0^{2-2x} x\left|\left|\frac{\partial T}{\partial x} \times \frac{\partial T}{\partial y}\right|\right| dxdy \\
&= \int_0^1 \int_0^{2-2x} \frac{7x}{2}\ dxdy \\
&= \int_0^1 7x - 7x^2\ dx \\
&= \frac{7}{6}.
\end{aligned}$$

Therefore, the x-coördinate of the center of mass is just the ratio of these two integrals, i.e. $\frac{7/6}{7/2} = \frac{1}{3}$.

Question 4. *Let T be the same triangle as in Question 3. Integrate the vector field $\mathbf{F}(x, y, z) = (xy, yz, zx)$ over the perimeter of this triangle, oriented in the counterclockwise direction as viewed from the positive octant.*

Solution. We could parametrize the boundary of this triangle, but that seems hard. Instead, we will use Stokes's theorem, which says that

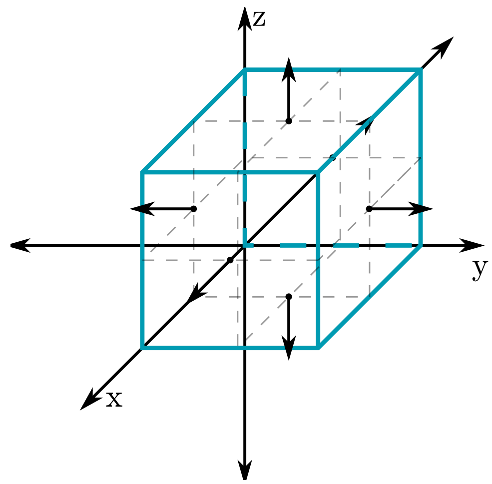
$$\int\limits_{\partial T} \mathbf{F} \cdot ds = \iint\limits_{T} \nabla \times \mathbf{F} \cdot dS.$$

Using this, we can instead integrate $\nabla \times \mathbf{F}$ over the triangle itself, because we already parametrized that! Convenient.

We do this here.

$$\begin{aligned}
\iint\limits_{T} \nabla \times \mathbf{F} \cdot dS &= \int_0^1 \int_0^{2-2x} (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial T}{\partial x} \times \frac{\partial T}{\partial y} \right) dA \\
&= \int_0^1 \int_0^{2-2x} \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right) \cdot \left(\frac{\partial T}{\partial x} \times \frac{\partial T}{\partial y} \right) dA \\
&= \int_0^1 \int_0^{2-2x} (0 - y, 0 - z, 0 - x) \Big|_{T(x,y)} \cdot \left(3, \frac{3}{2}, 1 \right) dA \\
&= \int_0^1 \int_0^{2-2x} -3y - \frac{3}{2} \left(3 - 3x - \frac{3}{2} y \right) - x \; dA \\
&= \int_0^1 \int_0^{2-2x} -\frac{9}{2} - \frac{3}{4} y + \frac{7}{2} x \; dA \\
&= \int_0^1 -9 + 9x - \frac{3}{8} (2 - 2x)^2 + 7x - 7x^2 \; dA \\
&= \int_0^1 -\frac{17}{2} x^2 + 19x - \frac{21}{2} \; dA \\
&= -\frac{17}{6} + \frac{19}{2} - \frac{21}{2} = -\frac{23}{6}.
\end{aligned}$$

Question 5. *Directly calculate the integral of $F(x, y, z) = (3x^2 y, -3xy^2, z)$ over the surface of the unit cube, using the orientation depicted below. Then, use the divergence theorem to calculate this in a much faster manner.*



Solution. If we want to do this directly, break the unit cube into its six sides

$$[0,1]\times[0,1]\times\{0\},[0,1]\times[0,1]\times\{1\},$$
$$[0,1]\times\{0\}\times[0,1],[0,1]\times\{1\}\times[0,1]$$
$$\{0\}\times[0,1]\times[0,1],\{1\}\times[0,1]\times[0,1],$$

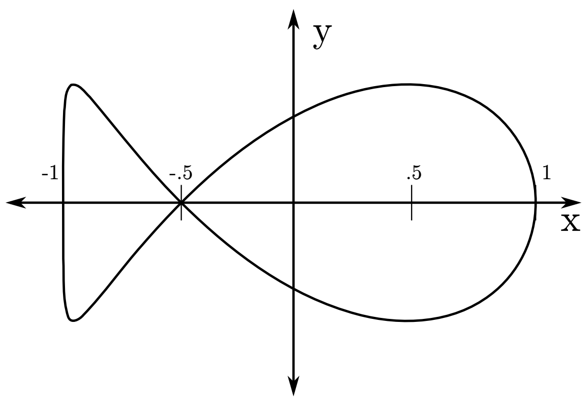
notice that the normals to these sides are precisely the normals $(0,0,\pm1),(0,\pm1,0),(\pm1,0,0)$ as depicted in the above diagram, and calculate

$$\begin{aligned}
&\iint_{\text{surface of cube}} F\cdot dS\\
=&\int_0^1\int_0^1 F\big|_{(x,y,0)}\cdot(0,0,-1)dxdy+\int_0^1\int_0^1 F\big|_{(x,y,1)}\cdot(0,0,1)dxdy\\
&\quad+\int_0^1\int_0^1 F\big|_{(x,0,z)}\cdot(0,-1,0)dxdz+\int_0^1\int_0^1 F\big|_{(x,1,z)}\cdot(0,1,0)dxdz\\
&\quad+\int_0^1\int_0^1 F\big|_{(0,y,z)}\cdot(-1,-0,0)dydz+\int_0^1\int_0^1 F\big|_{(1,y,z)}\cdot(1,0,0)dydz\\
=&\int_0^1\int_0^1 0dxdy+\int_0^1\int_0^1 1dxdy+\int_0^1\int_0^1 0dxdz+\int_0^1\int_0^1 -3x\ dxdz\\
&\quad+\int_0^1\int_0^1 0dydz+\int_0^1\int_0^1 3ydydz\\
=&1.
\end{aligned}$$

Alternately, if you use the divergence theorem, we can calculate this in a much faster way:

$$\iint\limits_{\text{surface of cube}} F \cdot dS = \iiint\limits_{\text{cube}} (\text{div} F) dV$$
$$= \int_0^1 \int_0^1 \int_0^1 (6xy - 6xy + 1) dx dy dz$$
$$= \int_0^1 \int_0^1 \int_0^1 1 dx dy dz = 1.$$

Question 6. *Let* $c(t) = \left(\cos(t) - \frac{\sin^2(t)}{2}, \cos(t)\sin(t)\right)$ *denote the "fish curve" drawn below:*



Find the area contained within this curve.

Solution. This looks like a textbook example of when to use the Green's theorem formula for the area contained in a curve. Specifically, Green's theorem, as applied to finding the area contained within a curve, says that if a region R is bounded by some simple closed curve $c(t)$ that is oriented positively (i.e. so that R is on the left as we travel along $c(t)$), then

$$\text{area}(R) = \iint\limits_R 1 dx dy \overbrace{=}^{\text{Green's theorem}} = \frac{1}{2} \int_{c(t)} (-y, x) \, dc.$$

If we just plug in our curve, we get that $\frac{1}{2}\int_{c(t)} (-y, x)\, dc$ is

$$\begin{aligned}
&\frac{1}{2}\int_0^{2\pi} \left(-\cos(t)\sin(t), \cos(t) - \frac{\sin^2(t)}{2}\right) \cdot \left(-\sin(t) - \sin(t)\cos(t), \cos^2(t) - \sin^2(t)\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\cos(t)\sin^2(t) + \cos^2(t)\sin^2(t) + \cos^3(t) - \cos(t)\sin^2(t) - \frac{\cos^2(t)\sin^2(t)}{2} + \frac{\sin^4(t)}{2}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{\cos^2(t)\sin^2(t)}{2} + \cos^3(t) + \frac{\sin^4(t)}{2}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{\sin^2(2t)}{8} + \cos(t)(1-\sin^2(t)) + \frac{(1-\cos(2t))^2}{8}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{1-\cos(4t)}{16} + \cos(t)(1-\sin^2(t)) + \frac{1-2\cos(2t)+\cos^2(2t)}{8}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{1-\cos(4t)}{16} + \cos(t)(1-\sin^2(t)) + \frac{1-2\cos(2t)}{8} + \frac{1+\cos(4t)}{16}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{1}{4} + \cos(t)(1-\sin^2(t)) - \frac{\cos(2t)}{4}\right) dt \\
=&\frac{1}{2}\left(\frac{t}{4} + \sin(t) - \frac{\sin^3(t)}{3} - \frac{\sin(2t)}{8}\right)\Big|_0^{2\pi} \\
=&\pi/4.
\end{aligned}$$

But is this plausible? Well: looking at our fish curve, it seems to contain about (in the head-part) the area of an ellipse from $-.5$ to 1 with y-height from -1 to 1, which is about $3\pi/2$. This is much greater than $\pi/4$, the area of a circle with radius .5. So: something has gone wrong!

What, specifically? Well, to apply Green's theorem, we needed a **simple closed curve** that was positively oriented. Did we have that here? No! In fact, our curve c has a self-intersection: $c(\pi/2) = c(3\pi/2)$, and in fact the tail part of our curve is oriented negatively (i.e. if we travel around our curve from $\pi/2$ to $3\pi/2$, our region is on the right-hand side. In fact, we've calculated the area of the head minus the area in the tail!

To calculate what we want, we want to take the integral above evaluated from $-\pi/2$ to $\pi/2$ (the head) and then add the integral from $3\pi/2$ to $\pi/2$ (travelling backwards here makes it so that we get the right orientation on the tail.) Specifically, we have

$$\begin{aligned}
\frac{1}{2}\left(\frac{t}{4} + \sin(t) - \frac{\sin^3(t)}{3} - \frac{\sin(2t)}{8}\right)\Big|_{-\pi/2}^{\pi/2} =&\frac{1}{2}\left(\frac{\pi}{8} - \frac{-\pi}{8} + 1 - (-1) + \left(-\frac{1}{3}\right) - \frac{1}{3} + 0 - 0\right) \\
=&\frac{\pi}{8} + \frac{2}{3},
\end{aligned}$$

while

$$\begin{aligned}
\frac{1}{2}\left(\frac{t}{4} + \sin(t) - \frac{\sin^3(t)}{3} - \frac{\sin(2t)}{8}\right)\Big|_{3\pi/2}^{\pi/2} =&\frac{1}{2}\left(-\frac{\pi}{8} - \frac{\pi}{8} + 1 - (-1) + \left(\frac{-1}{3}\right) - \frac{1}{3} + 0 - 0\right) \\
=&-\frac{\pi}{8} + \frac{2}{3};
\end{aligned}$$

therefore, our total area is $\frac{\pi}{8} + \frac{2}{3} + -\frac{\pi}{8} + \frac{2}{3} = \frac{4}{3}$.