Math 94	Professor: Padraic Bartlett

Practice GRE Exam!

Due Thursday, Week 10 — *UCSB 2015*

I wrote a practice GRE! It's a bit shorter (i.e. 40 problems in place of the normal 66) than a normal GRE exam, but it should still be a good warmup. Try it out! Give yourself about 2 hours or so to take this test, as it's a bit less than two-thirds the length of a normal GRE exam.

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1. Consider the sequence

$$\{\alpha_n\}_{n=1}^{\infty} = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$$

Which of the following sets of numbers is the collection of all values β for which a subsequence of $\{\alpha_n\}_{n=1}^{\infty}$ converges to β?

(a) $\mathbb{Q}$.
(b) $\mathbb{Q} \cap [0,1]$.
(c) $\mathbb{R}$.
(d) $[0,1]$.
(e) $\emptyset$.

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2. Suppose that a function $f : \mathbb{C} \to \mathbb{C}$ has a complex derivative at every point in $\mathbb{C}$. Which of the following must be true?

I. f can be written as a power series that converges on all of $\mathbb{C}$.
II. The integral of f along any closed path is 0.
III. f is a surjection from $\mathbb{C}$ to $\mathbb{C}$.

(a) I only.
(b) II only.
(c) II and III only.
(d) I and II only.
(e) I, II and III.

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3. Suppose that $f(x)$ has Taylor series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$, and that $g(x) = (f(x))^2$. What is $g'''(0)$?

(a) $2(a_0 + a_1 + a_2)$.
(b) $6(a_0a_3 + a_1a_2)$
(c) $3(a_1a_2 + a_0a_3)$.
(d) $12(a_0a_3 + a_1a_2)$
(e) Not enough information to tell.

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4. Suppose that G is a group and $g \in G$ is an element such that the set $\{g^{10}, g^6, g^{15}\}$ generates all of G. Which of the following numbers are **not** possible orders of G?

   (a) 1.
   (b) 7.
   (c) 12.
   (d) 30.
   (e) They are all possible orders for G.

5. An element r in a ring R is called **irreducible** if we cannot write r as the product of two non-unit elements. A ring R is called a **unique factorization domain** if we can write every element of r uniquely as some product of irreducible elements.

   Which of the following are unique factorization domains?

   I. $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$.
   II. $\mathbb{R}[x]$, the collection of all real-valued polynomials with coefficients in $\mathbb{R}$.
   III. $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

   (a) I only.
   (b) II only.
   (c) II and III only.
   (d) I and II only.
   (e) I, II and III.

6. Fix any real number a. How many solutions (v, w, x, y, z) are there to the following set of linear equations?

$$\begin{aligned} av + aw &= 1 \\ aw + ax &= 1 \\ ax + ay &= 1 \\ ay + az &= 1 \\ aw + az &= 1 \end{aligned}$$

   (a) No solutions exist.
   (b) Exactly one solution exists.
   (c) Exactly two solutions exist.
   (d) Infinitely many solutions exist.
   (e) There is not enough information to answer this problem.

7. Let S be the surface given by the level set $x^2 + y^3 + z^4 = 3$. Which of these planes are tangent to S at the point $(1, 1, 1)$?

   (a) $x = z$.

   (b) $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = \frac{13}{12}$.

   (c) $2x + 3y + 4z = 3$.

   (d) $9 = x + y + z$.

   (e) None of these are the tangent plane.

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8. Let $g : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by the equation $(a, b, c, d) = (a^b, c^d)$, and $h : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by the equation $h(x, y) = (x, y, xy, x)$.

   Find the derivative of $(g \circ h)(x, y)$ with respect to x, evaluated at $(x, y) = (1, 1)$.

   (a) (0,e).

   (b) (1,1).

   (c) $(e, 1)$.

   (d) $(1, e^2)$.

   (e) Undefined.

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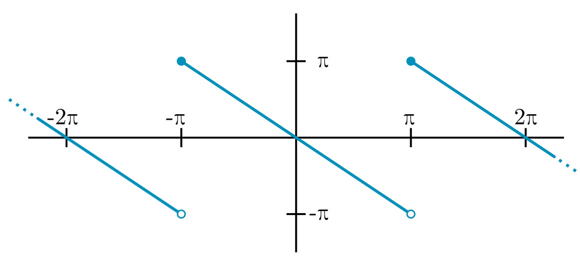
9. Which of the following Fourier series corresponds to the **sawtooth wave** $s(x)$ drawn below?

$$s(x) = \begin{cases} x, & x \in [-\pi, \pi), \\ s(x \pm 2\pi), & \forall x \in \mathbb{R}. \end{cases}$$



   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi(-1)^{n+1} \cdot \cos(nx)}{n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi \sin(nx)}{2^n}$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx) x^n}{n!}$.

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi(-1)^{n+1} \cdot \sin(nx)}{2n}$.

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx) + \cos(nx)}{n^2}$.

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10. Suppose that $\langle F, +, \cdot \rangle$ is a field. Which of the following properties **must not** be true?

(a) F is finite.
(b) $\forall x \in F, \exists y \in F$ with $x \cdot y = 1 + 1$.
(c) $\exists x, y \in F$ with $xyx^{-1}y^{-1} \neq 1$.
(d) $\mathbb{Q}$ is a subset of F.
(e) If G is a sobgroup of $\langle F^{\times}, \cdot \rangle$, then G is cyclic.

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11. Suppose that $f(x)$ is a function such that $f'(x) = \cos(2x)e^{3x}$, and $f(0) = 0$. What is $f(x)$?

(a) $\frac{2}{9}\sin(2x)e^{3x} + \frac{2}{9}\cos(2x)e^{3x} - \frac{2}{9}$.
(b) $\frac{1}{36}\sin(2x)e^{3x}$.
(c) $\frac{3}{5}\sin(2x)e^{3x} + \frac{2}{5}\cos(2x)e^{3x} - \frac{3}{5}$.
(d) $\frac{3}{13}\cos(2x)e^{3x} + \frac{2}{13}\sin(2x)e^{3x} - \frac{3}{13}$.
(e) $\frac{5}{3}\sin(2x)e^{3x} + \frac{10}{9}\cos(2x)e^{3x} - \frac{5}{3}$.

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12. What is the maximum value of $f(x, y) = 4x - 3y$ given the constraint $x^2 + y^2 = 4$?

(a) 1. (b) 2. (c) 6. (d) 8. (e) 10.

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13. Look at the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ q, & x = \frac{p}{q}, p, q \in \mathbb{Z}, gcd(p,q) = 1 \end{cases}$$

Let C denote the collection of all points at which $f(x)$ is continuous. Which of the following properties hold for C?

I. C is open. II. C is closed. III. C is compact.

(a) I only.
(b) II only.
(c) I and II only.
(d) II and III only.
(e) I, II and III.

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14. Find the integral

$$\int_0^1 \frac{x^{1/2}}{1 + x^{1/3}} dx.$$

(a) $\frac{1}{2}$.
(b) $5\left(\frac{1}{5} + \frac{1}{3} - \frac{\pi}{4}\right)$
(c) $\pi/4 - \frac{1}{5} + \frac{1}{7}$
(d) $6\left(\frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{\pi}{4}\right)$
(e) Does not exist.

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15. Consider the process of rolling a fair n-sided die. Let X be the random variable corresponding to the face that appears when such a die is rolled. What is the standard deviation of X?

(a) n.
(b) $\sqrt{\frac{n^2 - 1}{12}}$.
(c) $\binom{n}{2}$.
(d) $\sqrt{\frac{n+1}{2}}$.
(e) $\frac{6}{\sqrt{(n+1)(2n+1)}}$.

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16. Suppose that A is a $n \times n$ matrix with 2's on the diagonal and 1's elsewhere. What is the characteristic polynomial of A?

(a) $(x-2)^n$.
(b) $(x-1)^{n/2}(x-2)^{n/2}$.
(c) $x \cdot (x-n+1)^{n-1}$.
(d) $x^n - nx + 2$.
(e) $(x-1)^{n-1}(x-n-1)$.

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17. Which of the statements below is **not** equivalent to the logical formula

$$((A \vee B) \wedge (A \to C) \wedge (B \to C)) \to C \quad ?$$

(a) $(A \to B) \Leftrightarrow (\neg B \to \neg A)$
(b) $((\neg A \to B) \wedge (\neg A \to \neg B)) \to A$
(c) $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$
(d) $((A \to B) \vee (B \to A) \vee (A \to C) \vee (C \to A)) \to C$
(e) $((A \to B) \wedge (B \to C)) \to (A \to C)$

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18. What is the integral of the vector field $F(x,y,z) = (x,y,z)$ over the outward-oriented surface S formed by intersecting the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$ with the positive half-space $\{(x,y,z) \mid z \geq 0\}$?

(a) $\frac{8\pi}{3}$.
(b) $-\frac{8\pi}{3}$.
(c) 16.
(d) 0.
(e) 8π.

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19. Suppose that A is a $n \times n$ matrix with n distinct eigenvalues, and let $\vec{x}$ denote the vector $(1,2,3,\ldots n)$. Suppose that for any permutation $\sigma \in S_n$, we have $\lim_{n\to\infty} A^n \cdot \sigma(\vec{x}) = \vec{0}$. What must always be true about A?

(a) A is nilpotent.
(b) More than half of the entries of A are zero.
(c) A has no nonzero eigenvalues.
(d) All of A's eigenvectors are in $(-1,1)$.
(e) A is invertible.

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20. What is the length of the curve $\gamma(t) = (t, \ln(\sec(t))$ on the interval $[0, \pi/4]$?

(a) $\ln(\sqrt{2}/2 + 1)$.
(b) $\ln(\sqrt{2} + 1)$.
(c) $\ln(\sqrt{2}/2)$.
(d) $\pi \cdot \ln(\sqrt{2})$.
(e) $1 - \ln(\sqrt{2})$.

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21. Which of the following five objects has a bijection with $\mathbb{R}$?

(a) The Gaussian integers.
(b) The collection of all finite subsets of $\mathbb{N}$.
(c) The collection of all co-finite subsets of $\mathbb{Z}$.
(d) The collection of all ordinal numbers.
(e) The set of all uncountable subsets of the real line.

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22. Let $f(x) = y$ be the solution to the the differential equation

$$\frac{dy}{dx} + 3y = x^2 + 1,$$

given the boundary condition that at $x = 0$ we want $y = 0$.
What is $f(0)$?

(a) 0.
(b) 1/3.
(c) 5/9.
(d) 6/7.
(e) $-2/3$.

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23. How many roots does the equation $2x^3 + 3x^2 + 6x + c = 0$ have?

(a) 0.
(b) 1.
(c) 2.
(d) 3.
(e) It depends on c.

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24. Find the volume of the tetrahedron spanned by the three vectors $(1, 2, 3), (2, 3, 1), (3, 1, 2)$.

(a) 6.
(b) 9.
(c) 12.
(d) 18.
(e) 36.

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25. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$. II. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$. III. $\sum_{n=1}^{\infty} \frac{1}{n \ln(n)}$.

(a) I and II only.
(b) I and III only.
(c) II and III only.
(d) I, II and III.
(e) None of the above.

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26. As n goes to infinity, which of the five expressions listed below is the **largest**?

(a) The number of degree-n polynomials with coefficients in $\{1, \ldots n\}$.
(b) The number of even-order permutations in S_n.
(c) The number of subsets of the set $\{1, 2 \ldots n\}$.
(d) The reciprocal of the coefficient of x^n in e^x's Taylor series.
(e) $\binom{2n}{n}$.

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27. Suppose that G is a set endowed with a binary operation $\cdot : G \times G \to G$, such that the following properties hold:

- For all $a, b, c \in G$, we have $(a \cdot (b \cdot c)) = (a \cdot b) \cdot (a \cdot c)$.
- There is an element $e \in G$ such that for all $a \in G$, $e \cdot a = a \cdot e = a$.

Which of the following properties must be true?

I. G is a group.
II. G is infinite.
III. For any $g \in G$, there is some $n \in \mathbb{N}$ such that $g^n = e$.

(a) II only.
(b) III only.
(c) I and III only.
(d) I, II and III.
(e) None of the above.

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28. What is the average value of the function $f(x, y) = xy$ over the set $S = \{(x, y) \mid x^2 + y^2 = 1, x, y \geq 0\}$ of points on the unit circle with positive coördinates?

(a) $1/2\pi$. (b) 1. (c) $1/\pi$. (d) 0. (e) $2\pi/3$.

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29. Consider the following algorithm, that takes in as input two positive natural numbers n, m and outputs another natural number p:

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input (n)
input (m)
p:= 0

while n > 0
        begin
                if (n mod 2 == 1)
                        p := p + m
                        n := (n-1)/2
                        m:= 2*m
                else
                        n := n/2
                        m := 2*m
        end
output(p)
```

On input $n = 100, m = 2$, what is the output of this process?

(a) 0. (b) 2. (c) 50. (d) 100. (e) 200.

30. Let A be a $n \times n$ matrix. Which of the following statements is false?

(a) The dimension of the null space of A plus the dimension of the kernel of A is n.

(b) The dimension of the row space of A is the dimension of the image of A.

(c) The dimension of the column space of A is never less than the dimension of the row space of A.

(d) The rank of A plus the dimension of the null space of A is at least the dimension of the row space.

(e) The sum of the dimensions of A's row, column and null spaces is always less than $2n$.

31. Determine the radius of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n) x^n}{n}$$

(a) ∞.
(b) 0.
(c) 1.
(d) π.
(e) n.

32. Suppose that A, B, C are three pairwise independent events. Which of the following **is not necessarily** true?

(a) $P(A) = P(A|B)$.
(b) $P(A) \cdot P(B) \cdot P(C) = P(A \cap B \cap C)$.
(c) $P(A) = \dfrac{P(A \cap C)}{P(C)}$.
(d) $(P(A) \cdot P(B) \cdot P(C))^2 = P(A \cap B) \cdot P(A \cap C) \cdot P(B \cap C)$.
(e) $P(B \cap C) = P(B) \cdot P(C)$.

33. If $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ is the vector $(2, 0)$, what is x?

(a) 0.
(b) 1/3.
(c) 1.
(d) 2/3.
(e) 3.

34. Find the limit as x goes to 0 of the following fraction:

$$\frac{6\sin(x) - 6x + x^3}{x^5}.$$

(a) 0.
(b) 1.
(c) 1/20.
(d) 6/20.
(e) $-1/120$.

35. Given a set S, let $\mathcal{F}(S)$ denote the collection of all finite subsets of S, and $\mathcal{I}(S)$ denote the collection of all infinite subsets of S. Which of the following are true?

I. $\mathcal{F}(\mathbb{N})$ is a topology on the natural numbers.
II. $\mathcal{I}(\mathbb{N})$ is a topology on the natural numbers.
III. The union $\mathcal{F}(\mathbb{N}) \cup \mathcal{I}(\mathbb{N})$ is a topology on the natural numbers.

(a) I only.
(b) II only.
(c) III only.
(d) II and III only.
(e) I, II and III only.

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36. We call a group G "pleasant" if it is abelian and every non-identity element g in G has prime order. Up to isomorphism, how many pleasant groups are there of order at most 25?

(a) 0.
(b) 9.
(c) 16.
(d) 25.
(e) 31.
(f) Infinitely many.

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37. What is the antiderivative of

$$\frac{x-1}{x^2+x} \ ?$$

(a) $\dfrac{\tan(x)}{x}$.
(b) $\ln(x^2+x)$.
(c) $2\ln(x+1) - \ln(x)$.
(d) $\dfrac{1}{x^2-x}$.
(e) $\dfrac{\cos(x)}{x\sin(x)}$.

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38. A sequence of numbers $d_1, \ldots d_n$ is called **graphic** if there is a simple undirected graph G on n vertices $\{v_1, \ldots v_n\}$ where the degree of each vertex v_i is equal to d_i.

Which of the following sequences are graphic?

I. $d_1 = 1, d_2 = 2, d_3 = 3, d_5 = 4, d_5 = 5$.
II. $d_1 = 1, d_2 = 1, d_3 = 2, d_5 = 2, d_5 = 3$.
III. $d_1 = 3, d_2 = 3, d_3 = 3, d_5 = 3, d_5 = 3$.

(a) I only.
(b) III only.
(c) II and III only.
(d) I and II only.
(e) None of the above.

39. Find 2015^{2015} mod 7.

(a) 0. (b) 1. (c) 2. (d) 4. (e) 6.

40. Which of these trigonometric expressions is equal to $\cot(x) + \tan(x)$?

(a) $\cos(x)\sin(x) - \sin(x)\cos(x)$.
(b) $\sin^2(x) - \cos^4(x)$.
(c) $\tan(x)\sin(x)$
(d) $\tan(x) - \csc(x)$.
(e) $\sec(x)\csc(x)$.