Math 7H	Professor: Padraic Bartlett

Lecture 7: Electrical Networks and Random Walks II

Week 7	*UCSB*

In our last class, we built some nice correspondences between random walks and electrical networks, as given by the following theorem:

Theorem. Suppose that we have a connected graph G with edges weighted by some labeling w_{xy}. Define a **drunkard's walk** starting at a vertex x in our graph as the following process:

- Initially, the drunkard starts at x.
- Every minute, if a drunkard is at some vertex z, it randomly chooses one of the elements $y \in N(z)$ with probability given by the weights on its edges– i.e. each neighbor has probability w_{zy}/w_z of being picked – and goes to that vertex.

Let a, b be a pair of distinguished vertices in our graph, and $p(x)$ be the probability that a drunkard starting at the vertex x will make it to vertex b before vertex a.

Then $p(x) = v(x)$, if we turn our graph G into a electrical network with a connected to ground, a unit of electrical potential sent across a and b, and replace every edge $\{x, y\}$ of G with a resistor with conductance w_{xy}.

This gives us an excellent interpretation of voltage in terms of our random walk. This might lead us to wonder if other properties, like current, also correspond nicely to properties about random walks! To establish this, we first prove the following lemma:

Lemma 1. *Take a weighted graph with two distinguished vertices A, B on which we are trying to model a random walk: on this graph, a walker starts at A, and at each time step randomly (using our edge weighting) pick a neighbor of the vertex that it is currently at and wanders to that vertex. If it ever reaches B it stops; otherwise, it continues (unlike in our previous interpretation, where the walker stopped if it arrived at some other home vertex H as well.) Let $u(x)$ denote the average number of times that a walker will wander through the vertex x before reaching B in this model.*

Consider the corresponding circuit to this weighted graph, where all edges have conductance corresponding to our weights, B is grounded, and a potential of $\frac{u(A)}{C_A}$ volts is established between B and A. Then we have the following relation between $v(x)$, the voltage, and $u(x)$:

$$v(x) = \frac{u(x)}{C_x}.$$

Proof. Trivially, we have $v(B) = 0 = u(B)/C_B$, because B is grounded and no vertex can visit B before, um, visiting B. As well, we've specifically asked that $v(A) = u(A)/C_A$ in our problem's statement.

Finally: for any vertex $x \neq A, B$, we have

$$u(x) = \sum_{y \in N(x)} (\text{chance of going from y to x}) \cdot u(y);$$

i.e. the number of times we'll go through x is just the sum of the number of times we'll visit any of the neighbors of x, weighted by the chances of subsequently going from y to x.

From our above discussion, we know that the chance of walking from y to x is $\frac{C_{xy}}{C_y}$. If we use this observation and do a little symbolic manipulation, we can get the following:

$$\begin{aligned} u(x) &= \sum_{y \in N(x)} \frac{C_{xy}}{C_y} \cdot u(y) \\ &= \sum_{y \in N(x)} \frac{C_{xy} \cdot C_x}{C_y \cdot C_x} \cdot u(y) \\ \Rightarrow \qquad \frac{u(x)}{C_x} &= \sum_{y \in N(x)} \frac{C_{xy}}{C_x} \cdot \frac{u(y)}{C_y} \end{aligned}$$

So: we've seen these linear equations before! In particular, if we replace the $\frac{u(z)}{C_z}$'s with $v(z)$'s, these are precisely the linear equations that we asked the voltage function $v(x)$ to satisfy. We proved that these equations have a unique solution; therefore, this forces $v(x)$ and $\frac{u(x)}{C_x}$ to be equal, as claimed. $\square$

How does this relate to current? Well: in a super-simplistic sense, we can model the current flowing through an oriented edge (x, y) as the "flow" of electrons from x to y: i.e. if we have electrons randomly bumbling about on our graph starting at b and wandering around until they get to a, we might hope that i_{xy} is the average number of electrons that go from x to y, minus the number that go "backwards" along this edge from y to x.

We prove this here:

Lemma 2. *Let G be a graph as above, with potential difference between the ground vertex B and the source vertex A still set to $u(A)/C_A$. Then we have the following relation:*

$$i_{xy} = u(x)\frac{C_{xy}}{C_x} - u(y)\frac{C_{xy}}{C_y}.$$

In other words, the current between x and y is just the average number of times a random walker "uses" the edge (x, y) to go from x to y, minus the average number of times a random walker walks backwards on said edge from y to x!

Proof. We simply calculate, using our identities:

$$i_{xy} = (v(x) - v(y)) \cdot C_{xy} = \left(\frac{u(x)}{C_x} - \frac{u(y)}{C_y}\right) C_{xy} = u(x)\frac{C_{xy}}{C_x} - u(y)\frac{C_{xy}}{C_y}.$$

$\square$

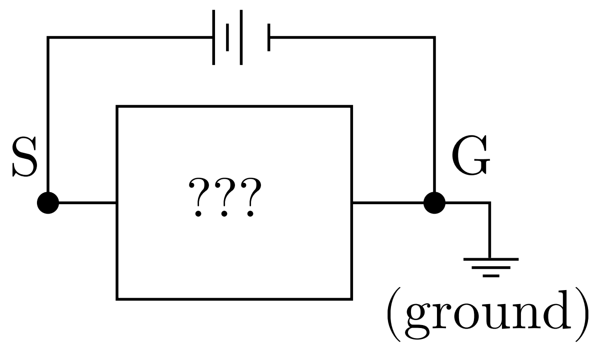
Electricity! Random walks! Apparently, mostly the same. So: why mention this in a graph theory class, other than the basic underlying structure? Consider the following puzzle posed by Polya (amongst others):

Question 3. *Suppose that you have placed a random walker placed at the origin of a d-dimensional integer lattice $\mathbb{Z}^d$, and let it wander. Given enough time, will the random walker return to the origin? Or is there a nonzero chance that the random walker will wander forever without returning to the origin?*

1 Circuits as Black Boxes

To attack this kind of question, it might help to introduce some new ideas. Specifically, suppose that we have a circuit with two points A, G, where we've grounded G and have a voltage of $1v$ established at A. If you have done this, then there is some amount of current flowing out of A. Denote this quantity as i_A, and note that i_A is given by the sum $\sum_{x \in N(A)} i_{Ax}$. Note that by Kirchoff's laws, the quantity of current that flows out of A is the same as the quantity of current that flows into G, because the sum of the currents through every vertex not A, B is equal to 0, and therefore whatever flows out of A must eventually flow into G.

Now, imagine simply covering up all of the connections and other bits between A and G with some sort of big black box. If we do this, then our circuit just looks like the following:



In this sense, we can simply "abstract" the rest of our circuit as some particularly large and bulky resistor, with effective resistance (which we denote R_{eff}) given by Ohm's law:

$$\frac{V(A) - V(G)}{i_A} = R_{\text{eff}}$$

Similarly, we can define $C_{\text{eff}} = 1/R_{\text{eff}}$.

Earlier, we noted that the current across an edge (x, y) was proportional to the expected number of paths from x to y minus the expected number of paths from y to x, up to scaling by the voltage we've established between the ground and the source. Does this idea still hold here? In other words: is there some connection between i_a and the total number of paths from A to G? Well: calculating, we have

$$\begin{aligned}
i_A &= \sum_{y \in N(A)} (v(A) - v(y)) \cdot C_{Ay} \\
&= \sum_{y \in N(A)} (v(A) - v(y)) \cdot \frac{C_{Ay}}{C_A} \cdot C_A \\
&= C_A \left(v(A) \sum_{y \in N(A)} \frac{C_{Ay}}{C_A} - \sum_{y \in N(A)} v(y) \frac{C_{Ay}}{C_A} \right) \\
&= C_A \left(v(A) - \sum_{y \in N(A)} v(y) \frac{C_{Ay}}{C_A} \right)
\end{aligned}$$

What is this quantity? Well:

- We know that $v(A)$ is 1, by assumption.
- We also know that $\frac{C_{Ay}}{C_A}$ denotes the probability that a random walker will travel from the vertex A to the vertex y.
- Finally, we know that $v(y) = p(y)$, the probability that a walk starting at y will make it to A before G.

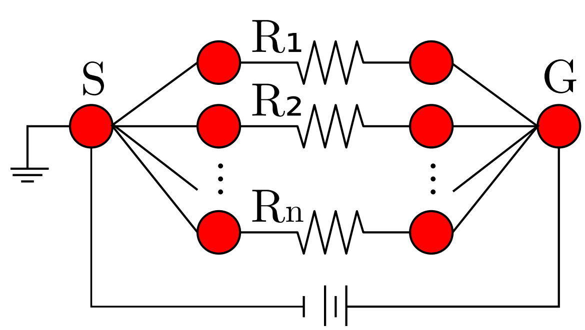
So: if we're starting at A and leaving to any of A's neighbors (which we pick with probability $\frac{C_{Ay}}{C_A}$), the chances of returning to A before making it to G is just $v(y)$. Therefore, the sum on the right inside of our parentheses is precisely the chances of starting at A and returning there before making it to G; consequently, 1 minus this sum is precisely the likelihood that we start at A and do not return there before wandering to G. Call this event, where we start at A and wander to G without returning to A, an "escape" event, and denote the probability of such an event happening p_{esc}.

If we plug this interpretation into our formula above, we get the following fantastically useful relation:

$$\frac{i_A}{C_A} = p_{\text{esc}}.$$

2 Resistance: Surprisingly Not Futile

This, basically, is almost the last tool we need to tackle Polya's problem on $\mathbb{Z}^d$. This is because we (for values of "we" that includes electrical engineers) know lots of techniques for finding effective resistances! In particular, suppose we have a series of resistors connected "in parallel," like in the picture below:

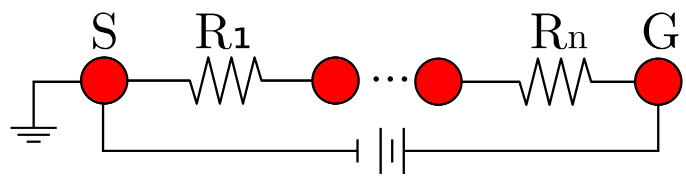


Then the effective resistance of the pictured circuit is the reciprocal of the sum of the reciprocals of the resistors:

$$\frac{1}{R_{\text{eff}}} = \sum_{i=1}^{n} \frac{1}{R_i}.$$

Alternately, you can think of this claim as the statement that the "effective conductance" of the circuit is the sum of the conductances of the circuit.

Similarly, suppose we have a circuit made of resistors linked in series, as depicted below:



Then the effective resistance of the pictured circuit is the sum of the resistors:

$$R_{\text{eff}} = \sum_{i=1}^{n} R_i.$$

It bears noting that you can deduce these properties from the two rules we've stated for electrical networks, Ohm's law and Kirchoff's law; the first property just says that the conductances sum when we have resistors in parallel, and the second says that resistances sum when we have resistors in series. We omit a formal proof here, but it's not very difficult (indeed, it's on your HW!)

The other property of electrical networks we're going to use throughout our proofs is Rayleigh's Monotonicity Theorem, which we state here:

Theorem 4. *If any of the individual resistances in a circuit increase, then the overall effective resistance of the circuit can only increase or stay constant; conversely, if any of the individual resistances in a circuit decrease, the overall effective resistance of the circuit can only decrease or stay constant.*

In specific, cutting wires (setting certain resistances to infinity) only increases the effective resistance, while fusing vertices together (setting certain resistances to 0) only decreases the effective resistance.

We also omit the proof of the statement here; it's trickier, but also doable / on the HW!

3 Random Walks in $\mathbb{Z}^d$

Given these tools, we are now equipped to tackle our question! Let's turn to $\mathbb{Z}^1$, as a quick warm-up. Our question, then, is whether a random walker starting at some point on the integer line (say the origin) will always return to the origin, or whether there's a nonzero chance that it wanders off forever.

All of our tools, as currently formulated, only apply to finite graphs. So, to study an infinite graph like $\mathbb{Z}^d$, we need to do the following:

- Let x be whichever node we're designating as the origin, and $G^{(r)}$ be the graph formed by taking all of the vertices connected to x by paths of length at most r.

- Turn this into a electrical network problem by soldering all of the vertices that are distance r from x together into one big ball (i.e. identifying all of these vertices together,) grounding them, putting one unit of voltage at x, and making all of the edges resistors with resistance 1. Then, via our earlier discussions, we can talk about the probability that a drunkard starting at x will make it to this point at distance r before returning to x. Denote this quantity as $p_{\text{esc}}^{(r)}$.

- Let p_{esc} be the limit $\lim_{r\to\infty} p_{\text{esc}}^{(r)}$. If this is nonzero, then there is some nonzero chance that our piffle will wander forever; if this is zero, then our piffle must eventually return to the origin.

- Notice that if it must eventually return to the origin, then it must eventually make it to any vertex w in G! This is because starting from the origin, we always have some nonzero chance to make it to w, and (because we return to the origin infinitely many times) we get infinitely many tries.

If G is a graph on which we return infinitely many times to the origin, we call G **recurrent**; if it is a graph where there is a chance that we will never return to the origin, we call G **transient**.

Theorem 5. *The one-dimensional lattice graph $\mathbb{Z}$ is recurrent.*

Proof. Let 0 be the origin, without any loss of generality. Using our earlier discussion, we know that

$$p_{\text{esc}}^{(r)} = \frac{i_0}{C_0} = \frac{1}{C_0} \cdot \frac{v(0)}{R_{\text{eff}}} = \frac{1}{C_0 R_{\text{eff}}}.$$

We know that the resistance of a string of r resistors in a row is r, from our earlier discussion about resistors in series. Consequently, because there are two such strings in parallel from the origin to distance r for any r, we know that their combined resistance is $\frac{1}{\frac{1}{r}+\frac{1}{r}} = \frac{r}{2}$. Therefore, because the conductance of the origin is $1+1=2$, we have

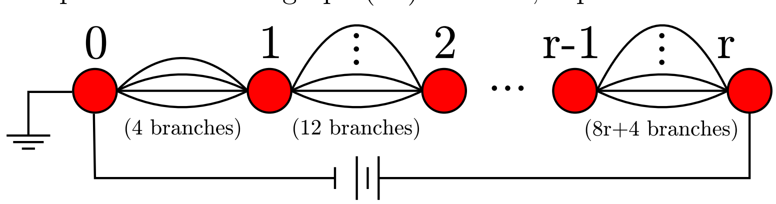
$$p_{\text{esc}}^{(r)} = \frac{1}{2 \cdot r/2} = \frac{1}{r}.$$

The limit as r goes to infinity of this quantity is 0; therefore, this walk is recurrent. □

Theorem 6. *The two-dimensional lattice graph $\mathbb{Z}^2$ is recurrent.*

Proof. Take our graph, turn it into an electrical network with origin $= (0,0)$, and perform the following really clever trick: for every r, let V_r be the collection of all of the vertices that are distance r from the origin under the taxicab metric (i.e. shortest length of a path.) Take our graph and **short** all of V_r's vertices into one huge clump, for each r: i.e. take the collection of all of the vertices at distance r, and just stick them all together! In essence, we are adding wires between all of the vertices at distance r with resistance 0, which (if you think of these wires not as connecting vertices that didn't use to be connected, but rather as replacing the wires of resistance "∞" between such vertices) is decreasing the resistance between certain vertices. We know that this reduces the overall resistance, because of Rayleigh's principle; therefore, we know that if this graph is recurrent, $\mathbb{Z}^2$ must be as well.

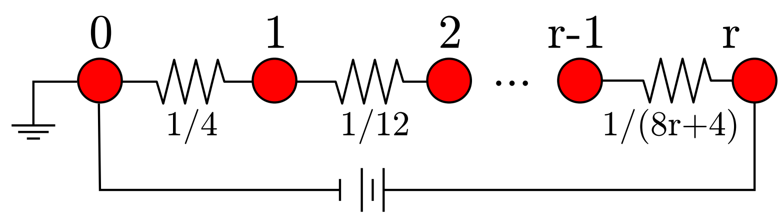
What does this process do to the graph $(\mathbb{Z}^2)^{(r)}$? Well, it produces the following picture:



Note that there are $8n + 4$ edges between the vertices at distance n and the vertices at distance $n + 1$, which we prove by induction:

- The base case is clearly true: there are 4 edges from the origin to the points $(\pm 1, 0)$, $(0, \pm 1)$ that are distance 1 from the origin.
- Now, notice that the collection of points at distance n forms a diamond, with corners given by the points $(\pm n, 0)$, $(0, \pm n)$ and side points given by points of the form (a, b) with $|a| + |b| = n$. There are 4 corner points and $4(n - 1)$ side points, an observation that you can either simply take as obviously true or as something we are proving by induction in the course of this proof.
- Each corner point has three edges connecting to points of distance $n+1$, and each side point has two edges connecting to points at distance $n + 1$. Therefore, by induction, we know that there are $3 \cdot 4 + 2 \cdot 4(n - 1) = 8n + 4$ edges to points at distance $n + 1$! (Of these edges at distance $n+1$, the new corner points each have one edge to a point at distance n, while the side points each have 2 edges to points at distance n. This, plus our observation that there are $8n + 4$ edges in total, tells us that there are 4 new corner points and $(8n + 4 - 4)/2 = 4n = 4((n + 1) - 1)$ new side points, if you were worried about that.)

What is the resistance here? Well: if there are $8n+4$ resistors between node n and node $n + 1$, we can regard our graph as equivalent to the path on $\{0, \ldots r\}$ where the resistance between vertices n and $n + 1$ is $\frac{1}{8n+4}$:



By adding these resistances together, we can finally calculate the effective resistance of this "shorted" $(\mathbb{Z}^2)^{(r)}$:

$$\sum_{i=1}^{r} \frac{1}{8i + 4}.$$

This sum diverges to infinity! Therefore, the current on these graphs, and thus the $p_{\text{esc}}^{(r)}$'s,must converge to 0. So $(\mathbb{Z}^2)^{(r)}$ is also recurrent. □

Lemma 7. *Suppose C is a circuit with two vertices x, y that are not connected by a resistor and are at the same potential: i.e. $v(x) = v(y)$. Then shorting together x and y does not change the voltages or currents in the circuit.*

Proof. Take x, y, and consider the "not-edge" between them as a resistor of infinite resistance. Notice that because $v(x) = v(y)$, we could change this resistance to anything we wanted without changing the current or voltages between these two vertices, as $v(x)-v(y) =$

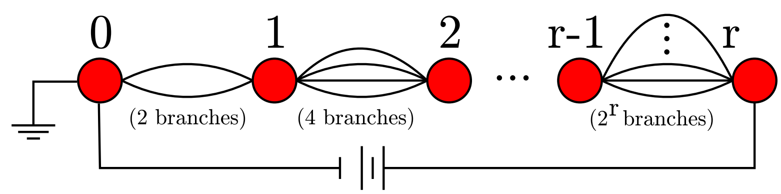
$i_{xy}R_{xy} = 0$, and so it doesn't matter what R_{xy} is! So, in particular, we can set R_{xy} to 0 and still have the same circuit/relations! (This is like noticing that if two variables a, b in a linear equation are equal, that we can replace all of the b's with a's!) $\square$

Theorem 8. *The three-dimensional lattice graph $\mathbb{Z}^3$ is transient.*

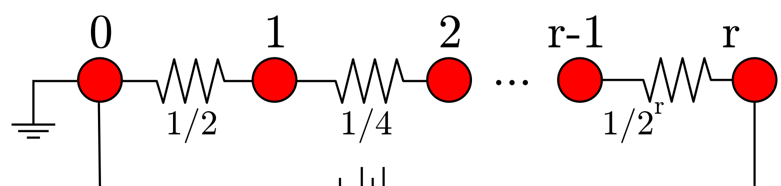
Proof. For $\mathbb{Z}^2$, the trick we used was to "short" a bunch of vertices together, and show that the resulting graph (which was simpler, even though its resistances were "lower") was recurrent. Here, in $\mathbb{Z}^3$, we're going to "cut" a number of resistors, and show that the resulting (simpler, higher-resistance) graph is transitive! (The normal proof of this theorem is much more difficult without these observations; it's only with this "shorting" and "cutting" that we can pull this off with such relative ease[1].)

In particular: lattices are **hard** to calculate resistances on. Let's try something simpler for a warm-up: a tree!

For example, let's consider the infinite binary tree graph T_2, where each edge has resistance 1, and we perform the standard trick of grounding everything at some cutoff distance r and put a potential of one volt at the root. Notice that (by symmetry) all of the nodes at any fixed distance k from the origin have the same potential: therefore, we can short them all together without changing the overall resistance of our circuit.



By using our earlier observations on resistors in parallel, we get that the above circuit is equivalent to the circuit below:



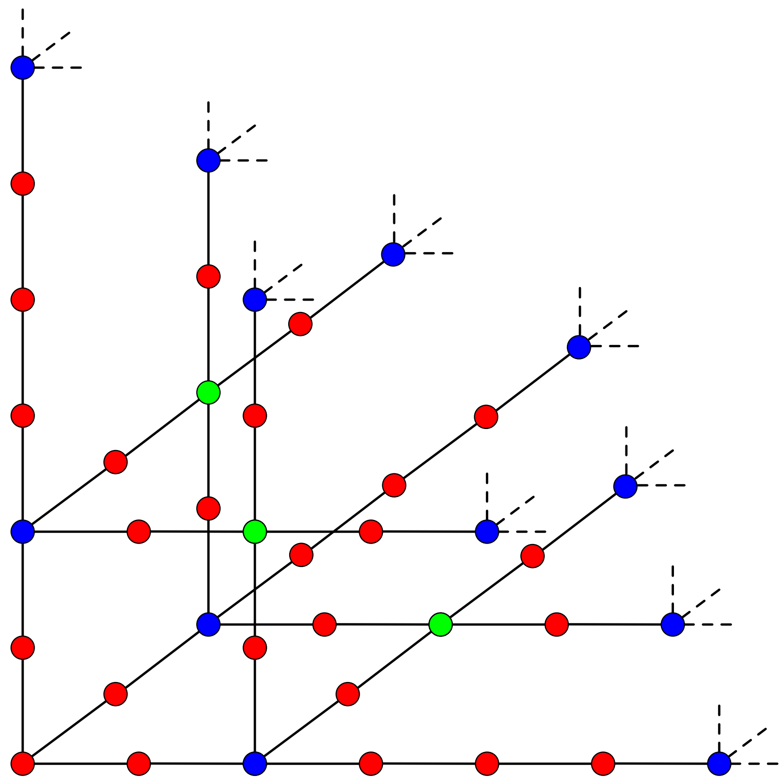
This has resistance $\sum 1/2^n = 1$.

Naively, we might hope that we can just find a copy of T_2 in $\mathbb{Z}^3$, and be done with our argument. However, the number of nodes at distance n from the root of T_2 is 2^n, while the number of nodes that are distance $\leq n$ from the origin in $\mathbb{Z}^3$ is $O(n^2)$: so we're not going to be able to nicely fit a binary tree in $\mathbb{Z}^3$! What will we do?

Answer: we will be **clever**. Specifically, let's stay with the tree structure. Binary, however, may have been overkill: perhaps the sum $\sum 1/2^n$ converges far faster than we need! Instead, we could aim for a tree who splits often enough that we'll get *some* sort of convergent thing at the end of the day, but not so fast that we can't fit it in $\mathbb{Z}^3$. (This seems like a plausible goal: things like the sum $\sum \frac{1}{n^2}$ converge, so we certainly don't need as much branching as the binary tree T_2.)

[1]Insert your own "short-cut" pun here.

To do this, consider the following kind of"tree:"



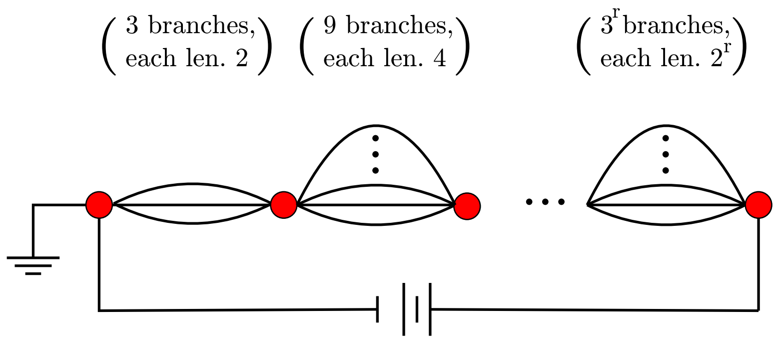
As currently drawn: not a tree. However, if you pretend that each of the green nodes are "doubled", by creating two vertices at each of those locations and passing only one branch through each node, it's a tree! Suppose for the moment that this picture is not lying to you: that the only overlapping parts of this tree are at the green vertices, and no branches or other such things overlap. Then, because the green nodes are at the same distances from the origin in the tree version of this graph, we know that they have the same voltage passing through them by symmetry — so there is no difference between the voltage/resistance/etc of the "tree" as drawn in our picture and the tree as realized by splitting the green nodes!

To give an explicit construction for the above picture: this tree is constructed by taking the positive octant of $\mathbb{Z}^3$ and starting from the origin. At the origin and each vertex with distance $\sum_{n=1}^{r} 2^n$ for every $r \geq 1$ (i.e. at the blue nodes,) our tree "branches" and creates three paths from these blue nodes: one branch that continues infinitely in the positive-x direction from that blue vertex, one that continues infinitely in the positive-y direction from that blue vertex, and one that continues infinitely in the positive-z direction from that blue vertex.

Notice that our tree only intersects at the "green" vertices in this picture, and specifically that these green vertices never coincide with one of these "blue" vertices. This is not hard to see: suppose that two tree branches managed to overlap at a blue vertex v that is distance $\sum_{n=1}^{r} 2^n$ from the origin. Then there must be two distinct blue vertices w_1, w_2 that we traveled from on distinct paths of length 2^r to get to v, both of which are distance $\sum_{n=1}^{r-1} 2^n$ from the origin. But this cannot happen: if we look at our two distinct paths of length 2^r, they are forming two sides of a square with side length 2^r in $\mathbb{Z}^3$ with v as one of its corners. Because the sum $\sum_{n=1}^{r-1} 2^n = 2^r - 2 < 2^r$, it is impossible for the two points w_1, w_2 to be distance $\sum_{n=1}^{r-1} 2^n$ from the origin and also be the two corners opposite v in this square.

Therefore our tree as drawn in $\mathbb{R}^3$ only overlaps at nodes that are not blue nodes, and therefore in particular only overlaps at vertices (i.e. it does not overlap on edges.) So, if we split it at each of these green nodes, we get an actual tree; moreover, because these vertices in the split tree all have the same voltages by symmetry, we can (again) see that there is no difference between the voltage/resistance/etc of the "tree" as drawn in our picture and the tree as realized by splitting the green nodes.

By identifying nodes of distance $\sum_{n=1}^{r} 2^n$ for every n from the origin, the graph on this tree restricted to the distances $\sum_{n=1}^{r} 2^n$ is equivalent to a circuit of the form



By applying our known results about resistors in series and parallel, we can see that the total resistance between any two nodes $n-1, n$ in the above circuit is

$$\frac{2^n}{3^n}.$$

Therefore, our tree at stage R has total resistance

$$\sum_{n=1}^{r} \frac{2^n}{3^n} = \frac{1-(2/3)^{r+1}}{1-(2/3)} - 1.$$

As r goes to infinity, this goes to 2; therefore, the current $i_A = v(A)/R_{\text{eff}} = 1/2$ at infinity is positive, and consequently the value $p_{\text{esc}} = i_A/C_A = \frac{1/2}{3} = 1/6$ is positive and nonzero. Therefore, by our earlier discussion, there is a nonzero chance of escape! In other words, our random walker may never return to the origin (and in fact, we've shown that they have at least a 1/6-th chance to do so!) $\square$

This gives us the following corollary, which is an excellent note to end our lecture on:

Corollary 9. *A lost drunkard will come home if and only if it cannot fly.*