Math 7H	Professor: Padraic Bartlett

Homework 1: Latin Squares

Due Tuesday, week 2, at the start of class — *UCSB 2015*

Try some of the problems below! As always, work on problems here until you've spent at least 90 minutes on this set. Show your work, so that I can see that you've spent time/effort on this!

This week's HW centers around the following idea:

Definition. A **partial latin square** of order n is a $n \times n$ array where each cell is filled with either blanks or symbols $\{1, \ldots n\}$, such that no symbol is repeated twice in any row or column.

Example. Here are a pair of partial 4×4 latin squares:

			4
2			
3	4		
4	1	2	

1			
	1		
		1	
			2

The most obvious question we can ask about partial latin squares is the following: when can we complete them into filled-in latin squares? There are clearly cases where this is possible: the first array above, for example, can be completed as illustrated below.

			4
2			
3	4		
4	1	2	

$\mapsto$

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

However, there are also clearly partial Latin squares that cannot be completed. For example, if we look at the second array

1			
	1		
		1	
			2

,

we can pretty quickly see that there is no way to complete this array to a Latin square: any 4×4 Latin square will have to have a 1 in its last column somewhere, yet it cannot be in any of the three available slots in that last column, because there's already a 1 in those three rows.

1. For any n, create a partial $n \times n$ Latin square that contains n symbols, that cannot be filled to a complete Latin square.

2. A partial Latin square is called **critical** if it has a completion to exactly one other Latin square. For example,

1	

   is a critical partial Latin square, as it can be completed to only one Latin square,

1	2
2	1

   .

   Find the smallest 4×4 critical Latin square.

3. (Open / my research!) For any constant $\epsilon > 0$, a partial $n \times n$ Latin square is called ϵ-sparse if it contains at most ϵn entries in any row or column, and also no symbol is used more than ϵn many times.

   Show that any 1/4-dense partial Latin square is completable.