(1) Find the solution of the initial-value problem

\[(1 + e^t)x \frac{dx(t)}{dt} = e^t, \quad x(0) = 1.\]

ANSWER: \[y(x) = \sqrt{2 \ln((1 + e^x)/2)} + 1.\]

(2) Find the general solution of

\[\frac{dx(t)}{dt} + 2tx(t) = 2te^{-t^2}.\]

ANSWER: \[x(t) = (t^2 + c)e^{-t^2}.\]

(3) Find the general solution of

\[3x^2 \frac{d^2y}{dx^2} + 11x \frac{dy}{dx} - 3y = 0, \quad x > 0.\]

ANSWER: \[y(x) = c_1x^{1/3} + c_2x^{-3}.\]

(4) Consider the equation

\[18 \frac{d^2x}{dt^2}(t) + 8 \frac{dx}{dt}(t) + 5x(t) = 18x''(t) + 8x'(t) + 5x(t) = 0.\]

(a) Find the solution satisfying the initial conditions \(x(0) = 1, \quad x'(0) = 3.\)

(b) Find constants \(A, a, b, \delta\) such that the solution found in (a) can be written in the form

\[x(t) = A e^{at} \cos(b(t - \delta/b)).\]

(c) Use part (b) to graph the solution found in (b) in the time interval \(t \in [0, 2\pi].\)

\[A : x(t) = \frac{\sqrt{173}}{2} e^{-t/4} \cos(t/2 - 1.418), \text{ with calculator: no needed for the final}.\]

(6) Find the solution of the differential equation \(x'' + 4x' + 4x = 1 + e^t\) satisfying the initial condition \(x(0) = 1, \quad x'(0) = 0.\)

(7) Given the function \(x(t) = Ae^{-t} + Be^{3t} + \cos(t) + 3\), find a second order linear constant coefficients differential equation for which this function is the general solution.

ANS: \[x'' - 2x' - 3x = -4\cos(t) + 2\sin(t) - 9.\]
(8) Consider the initial-value problem

\[
\begin{cases}
\frac{d^2x(t)}{dt^2} + 4x(t) = 4\cos(2t), \\
x(0) = 0, \quad x'(0) = 2.
\end{cases}
\]

(a) Find its solution \(x(t)\) (verify your answer).
(b) Sketch the graph of the solution for \(0 \leq t \leq 20\).
(c) If the function \(x(t)\) is the position at time \(t\) of a mass attached to a spring under the external force \(f(t) = 4\cos(2t)\), how do you describe the motion of the mass?

(9) Find the solution of the initial value problem

\[x''(t) - x'(t) - x(t) = \sin(x), \quad x(0) = 1, \quad x'(0) = -3.\]

Verify your answer.

(10) For the system of equations

\[
\begin{cases}
x'_1 = -x_1 + 2x_2, \\
x'_2 = -2x_1 - x_2,
\end{cases}
\]

(a) Find the general form of the solution (verify your answer).
(b) Find the solution satisfying the initial conditions \(x_1(0) = 1, \ x_2(0) = -1\).
(c) Determine whether the origin (stationary solution) is a source, spiral source, sink, spiral sink, center, or a saddle, (in the case of a spiral or a center decide if the solutions move clock-wise or counter clock-wise)

(11) For the system of equations

\[
\begin{cases}
x'_1 = 2x_1 - x_2, \\
x'_2 = 2x_1 + 2x_2,
\end{cases}
\]

(a) Find the general form of the solution (verify your answer).
(b) Find the solution satisfying the initial conditions \(x_1(0) = 1, \ x_2(0) = 0\).
(c) Determine whether the origin (the fixed point) is a source, sink, center, or a saddle.
(12) For the system of equations

\[ \begin{cases} 
  x'_1 = -x_1 + 2x_2, \\
  x'_2 = -2x_1 - x_2,
\end{cases} \]

(a) Find the general form of the solution (verify your answer).
(b) Find the solution satisfying the initial conditions \(x_1(0) = 1, \ x_2(0) = -1\).
(c) Determine whether the origin (stationary solution) is a source, spiral source, sink, spiral sink, center, or a saddle, (in the case of a spiral or a center decide if the solutions move clock-wise or counter clock-wise)

(13) For the system of equations

\[ \begin{cases} 
  x'_1 = -x_1 - x_2, \\
  x'_2 = x_1 + -3x_2,
\end{cases} \]

(a) Find the general form of the solution (verify your answer).
(b) Find the solution satisfying the initial conditions \(x_1(0) = 1, \ x_2(0) = -1\).
(c) Draw an approximate phase portrait.

(14) For the (inhomogeneous) system of equations

\[ \begin{cases} 
  x'_1 = 2x_1 - 3x_2 + t, \\
  x'_2 = x_1 - 2x_2 + 1,
\end{cases} \]

(a) Find the general form of the solution.
(b) Find the solution satisfying the initial conditions \(x_1(0) = 1, \ x_2(0) = 0\) (verify your answer).

(15) Find a 2 \times 2 matrix \(B\) having \(\lambda_1 = -1, \ \lambda_2 = 1\) as eigenvalues and

\[ \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

as the corresponding eigenvectors (verify your answer).
(16) Given the matrix
\[ A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \]
Find \( A^{1000} \) and \( B = \sqrt{A} \), i.e \( B^2 = A \) (verify your answer).

(17) For the system of equations
\[
\begin{align*}
x'_1 &= -x_1 + ax_2, \\
x'_2 &= -2x_2,
\end{align*}
\]
show that the origin \((0, 0)\) is an stable equilibrium solution regardless the value of the constant \(a\).

(18) For the system of equations
\[
\begin{align*}
x' &= 2x + 2y + 2x^2 \cos(xy), \\
y' &= 2x - y + xy^3,
\end{align*}
\]
classify the behavior of the solution near the stationary solution \((0, 0)\).

(19) Find the general solution of the in-homogeneous system
\[
\begin{align*}
x' &= -y - 2z + t, \\
y' &= -2y + z, \\
z' &= -z + t
\end{align*}
\]