Problem 1. Consider the function $f : [0, 1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 
0, & \text{if } x \notin \mathbb{Q} \cap [0, 1], \\
\frac{1}{q}, & \text{if } x = \frac{p}{q} \ (p, q \in \mathbb{Z}^+ \text{ relative primes}).
\end{cases}$$

(1)

Where is $f$ discontinuous? Using the definition prove that $f$ is Riemann integrable on $[0, 1]$. 
Problem 2. Consider the function $f : [0, 1] \to \mathbb{R}$ defined as

\[
(1a) \quad f(x) = \begin{cases} 
0, & \text{if } x \notin \mathbb{Q} \cap [0, 1], \\
1, & \text{if } x = \frac{p}{q}, \quad p, q \in \mathbb{Z}^+. 
\end{cases}
\]

Where is $f$ discontinuous? Using the definition prove that $f$ is not Riemann integrable on $[0, 1]$. 
Definition 1. A set $E \subseteq \mathbb{R}$ is said to be of “measure zero” if given $\epsilon > 0$ there is a countable collection of intervals $\{I_j\}_{j \in \mathbb{Z}^+}$ which covers $E$,

$$E \subseteq \bigcup_{j \in \mathbb{Z}^+} I_j \text{ such that } \sum_{j=1}^{\infty} |I_j| = \sum_{j=1}^{\infty} \text{length of } I_j < \epsilon.$$ 

Thus : (i) Prove that every countable set of $\mathbb{R}$ is a set of measure zero. and (ii) the countable union of sets of measure zero has measure zero.

Problem 3 Let $E$ be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Prove that $E$ is uncountable set of measure zero.
Definition 2. Let \( f : [a, b] \to \mathbb{R} \) be a bounded function, \( b - a < \infty \). For \( x \in [a, b] \) and \( \eta > 0 \) define

\[
\Omega(f, x, \eta) = \sup \{ |f(x_1) - f(x_2)| : x_1, x_2 \in (x - \eta, x + \eta) \cap [a, b] \},
\]
and the oscillation of \( f \) at a point \( x \in [a, b] \).

\[
\omega_f(x) = \lim_{\eta \to 0^+} \Omega(f, x, \eta) = \inf_{\eta > 0} \Omega(f, x, \eta).
\]

Problem 4 Prove that \( \omega_f(x) \) is defined for any \( x \in [a, b] \).
Problem 5 Prove that $f$ is continuous at $x_0$ if and only if $\omega_f(x_0) = 0$. 
Problem 6 Prove that for any $\mu > 0$ the set $A_\mu = \{ x \in [a, b] : \omega_f(x) \geq \mu \}$ is compact.
Problem 7 Prove that the set of discontinuities of \( f \) can be written as

\[
D_f = \bigcup_{j \in \mathbb{Z}^+} A_j = \{x \in [a, b] : \omega_f(x) \geq \frac{1}{j}\}.
\]
Problem 8 Prove that if for some $\epsilon > 0$, $\omega_f(x) < \epsilon$ for any $x \in [a,b]$, then there exists a $\eta > 0$ such that

$$\Omega(f, x, \eta) < \epsilon,$$

for any $x \in [a,b]$.

Hint: Use the compactness of $[a,b]$. 
At this point we are ready to prove the Lebesgue criterion for Riemann integrability.

**Theorem 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a bounded function and \( b - a < \infty \). Then \( f \) is Riemann integrable on \([a, b]\) if and only if the set of discontinuities of \( f \), \( D_f \) is a set of measure zero.

**Problem 9** Prove theorem 1.

**Hint:** Assume that \( f \) is RI and that \( D_f \) does not have measure zero. Prove that for some \( j_0 \in \mathbb{Z}^+ \)

\[ A_{j_0} = \{ x \in [a, b] : \omega_f(x) \geq \frac{1}{j_0} \} \]

is not a set of measure zero. Take a partition \( P \) of \([a, b]\), observe that the sub-intervals of this partition cover \( A_{j_0} \). Evaluate the difference between its upper and lower sum to get a contradiction.

Assume that \( D_f \) has measure zero. Then for any \( j \) the set \( A_j \) is compact and has measure zero. For \( j \) large one can cover \( A_j \) by a countable collection of intervals whose sum of their lengths is arbitrary small, say less than \( \epsilon/2 \). Show that one can expand these intervals to obtain a new collection of open intervals covering \( A_j \) and whose sum of their lengths is less than \( \epsilon \). Use the compactness of \( A_j \) to construct an appropriate partition, which allows to conclude the proof.
We saw in class that if $B$ is an open set of $\mathbb{R}$, then $B$ is the union of an at most countable collection of disjoint open intervals.

Let $B$ be an open bounded set of $\mathbb{R}$. Define $\chi_B$ by $\chi_B(x) = 1$, if $x \in B$, and $\chi_B(x) = 0$, if $x \notin B$.

$$\int \chi_B(x) \, dx = ?$$

Problem 10 Give an example of $B$ open and bounded such that $\chi_B$ is not Riemann integrable. Moreover, $\chi_B$ is not Riemann integrable even after any modification on a set of measure zero.