HOMEWORK #1 (due April 06 in class) Problems

–1) Let $E$ be a real vector space. Let $K$ be a convex subset of $E$ such that

(i) $K$ is absorbing, i.e.

\[ 0 \in K \quad \text{and} \quad \forall x \in E \, \exists \lambda > 0 \text{ such that } \lambda x \in K, \]

(ii) $K$ is symmetric, i.e. if $x \in K$, then $-x \in K$.

Define the functional $p_K : \to [0, \infty)$ as

\[ p_K(x) = \inf \{ a > 0 : x/a \in K \}. \]

Prove:

(a) \( \{ x \in E : p_K(x) < 1 \} \subseteq K \subseteq \{ x \in E : p_K(x) \leq 1 \} \),

(b) $p_K$ is a convex function,

(c) does $p_K$ defines a norm in $E$? a semi-norm?

–2) Let $p_1, p_2, p_3 \in [1, \infty]$ with $p_1 < p_2 < p_3$. Let $\theta \in (0, 1)$ such that

\[ \frac{1}{p_2} = \frac{\theta}{p_1} + \frac{1-\theta}{p_3}. \]

Prove that if $f \in L^{p_2} \cap L^{p_3}$, then $f \in L^{p_2}$ and

\[ \|f\|_{p_2} \leq \|f\|_{p_1}^{\theta} \|f\|_{p_3}^{1-\theta}. \]

–3) From Stein-Shakarchi “Real Analysis” problem 20 in Chapter 4, assuming that the sequence \( \{ f_n : n \in Z^+ \} \) is bounded instead that $\|f_n\| = 1$ for any $n \in Z^+$. Rephrase this results.

–4) Let $X$ be a Banach space and $X^*$ is dual. One says that the sequence \( (x_n)_{n \in Z^+} \subset X \) converges in the weak topology of $X$ (weakly in $X$) to $z \in X$ if for each $\psi \in X^*$

\[ \lim_{n \to \infty} \psi(x_n) = \psi(z) \quad \text{(in } \mathbb{C}). \]

Using that \( (L^p(\mathbb{R}^d))^* = L^q(\mathbb{R}^d), 1/p + 1/q = 1 \) for $p \in [1, \infty)$ prove:

(a) Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, define $f_n(x) = f(x-n)$, $n \in Z^+$. Prove that $f_n$ converges weakly in $X$ to 0. What can one say if $p = 1$?

(b) Let $d = 1$ and $f_n(x) = \sin(n|x|)$ if $|x| \leq \pi$ and $f_n(x) = 0$ if $|x| \geq \pi$ for each $n \in Z^+$. Prove that $f_n$ converges weakly in $L^p$, $1 < p < \infty$ to 0. What can one say if $p = 1$?

(c) Prove that if $H$ is a Hilbert space and $x_n$ converges weakly to $z$ such that $\|x_n\| \to \|z\|$, then $x_n$ converges in $H$ to $z$, i.e $\|x_n - z\| \to 0$ as $n$ tends to infinity.