\[ \text{HOMEWORK } \#2 \text{ (due April 20 in class)} \text{ Problems} \]

-1) Let \( X \) be a Banach space. Prove:
(i) If \( x \in X \) then
\[ \|x\| = \text{Sup}\{ |\varphi(x)| : \varphi \in X^*, \ |||\varphi||| \leq 1 \}. \]
(ii) If \( x \in X \), define \( \Phi_x : X^* \to \mathbb{C} \) as \( \Phi_x(\varphi) = \varphi(x) \). Prove that
\[ ||\Phi_x|| = ||x||. \]
Hence, \( X \hookrightarrow X^{**} \), continuous embedding.

-2) Let \( X \) be a Banach space. Let \( (x_n)_{n=1}^{\infty} \subset X \) such that \( x_n \) converges weakly to \( x^* \).
Prove that \( (x_n)_{n=1}^{\infty} \) is uniformly bounded, i.e. there exists \( M > 0 \) such that for any \( n = 1, 2, \ldots \), \( \|x_n\| \leq M \).

-3) Definition A : A function \( f \in L^p(\mathbb{R}) \) is \textit{differentiable in} \( L^p(\mathbb{R}) \) if there exists \( g \in L^p(\mathbb{R}^n) \) such that
\[ \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} - g(x) \right|^p \, dx \to 0 \quad \text{as} \quad h \to 0, \]
If such a function \( g \) exists (in this case it is unique) it is called the derivative of \( f \) in the \( L^p \)-norm.

Given \( f \in L^2(\mathbb{R}^n) \) prove that the following statements are equivalent:
(i) \( g \in L^2(\mathbb{R}) \) is the derivative of \( f \in L^2(\mathbb{R}) \) according to the Definition A.
(ii) There exists \( g \in L^2(\mathbb{R}) \) such that
\[ \int_{\mathbb{R}} f(x) \phi'(x) \, dx = - \int_{\mathbb{R}} g(x) \phi(x) \, dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}). \]
(iii) There exists \( \{f_j\} \subset C_0^\infty(\mathbb{R}) \) such that
\[ ||f_j - f||_2 \to 0 \quad \text{as} \quad j \to \infty, \]
and \( \{f_j\} \) is a Cauchy sequence in \( L^2(\mathbb{R}) \).
(iv) \( \xi \hat{f}(\xi) \in L^2(\mathbb{R}^n) \).
(v) \( \sup_{h>0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right|^2 \, dx < \infty. \)