ON THE PERSISTENCE PROPERTIES OF SOLUTIONS OF
NONLINEAR DISPERSIVE EQUATIONS IN WEIGHTED
SOBOLEV SPACES

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Abstract. We study persistence properties of solutions to some canonical dis-
persive models, namely the semi-linear Schrödinger equation, the $k$-generalized
Korteweg-de Vries equation and the Benjamin-Ono equation, in weighted Sobolev
spaces $H^s(\mathbb{R}^n) \cap L^2(|x|^l dx)$, $s, l > 0$.

1. Introduction

This work is concerned with persistence properties of solutions to some nonlinear
dispersive equations in weighted Sobolev spaces $H^s(\mathbb{R}^n) \cap L^2(|x|^l dx)$, $s, l > 0$. We
shall consider the initial value problems (IVP) associated to the following dispersive
models: the nonlinear Schrödinger (NLS) equation

$$i \partial_t u + \Delta u = \mu |u|^{a-1} u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad \mu = \pm 1, \quad a > 1,$$

the $k$-generalized Korteweg-de Vries ($k$-gKdV) equations

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad t, x \in \mathbb{R}, \quad k \in \mathbb{Z}^+,$$

and the Benjamin-Ono (BO) equation

$$\partial_t u + \mathcal{H} \partial_x^2 u + u \partial_x u = 0, \quad t, x \in \mathbb{R},$$

where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H} f(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y} dy = -i \text{sgn}(\xi) \hat{f}(\xi) \vee(x).$$

These models have been widely studied in several contexts. For example, the
KdV $k = 1$ in (1.2) was first deduced as a model for long waves propagating in
a channel. Subsequently the KdV and its modified form ($k = 2$ in (1.2)) were
found to be relevant in a number of different physical systems. Also they have
been studied because of their relation to inverse scattering theory [20]. The NLS
arises as a model in several different physical phenomena (see [61] and references
therein). In the particular, case $n = 1$ and $a = 3$ it has been shown to be completely
integrable [66]. The BO equation (1.3) was first deduced in [3] and [54] as a model
for long internal gravity waves in deep stratified fluids. It was also shown that it is
a completely integrable system (see [2], [12] and references therein).

We recall the notion of well posedness given in [34]: the IVP is said to be locally
well posed (LWP) in the function space $X$ if for each $u_0 \in X$ there exist $T > 0$ and
a unique solution $u \in C([-T, T] : X) \cap \ldots = Y_T$ of the equation, with the map data
$u_0$ solution being locally continuous from $X$ to $Y_T$. This notion of LWP includes
the “persistent” property, i.e. the solution describes a continuous curve on $X$. In
particular, it implies that the solution flow defines a dynamical system in $X$. When
$T$ can be taken arbitrarily large one says that the corresponding IVP is globally well posed (GWP) in $X$.

First, we shall study the Schrödinger equation (1.1).

## 2. The Schrödinger Equation (1.1)

The results in [9], [10], [21], [35], and [65] yield the following LWP theory in the classical Sobolev spaces $H^s(\mathbb{R}^n)$ for the IVP associated to the NLS equation (1.1).

### Theorem A
Let (2.2)

\[ T \equiv T(a) \]

be taken arbitrarily large one says that the corresponding IVP is globally well posed (GWP) in $X$.

\[ T \cap \mathbb{R}^n \]

First, we shall study the Schrödinger equation (1.1).

\[ \dot{u} + i\lambda u + \nabla_x \frac{q}{p} D^s_{x,t} u = 0, \quad u(0) = u_0 \in H^s(\mathbb{R}^n) \]

### Notations
(a) for $1 < p < \infty$ and $s \in \mathbb{R}$

\[ L^p(\mathbb{R}^n) = (1 - \Delta)^{-s/2} L^p(\mathbb{R}^n) = J^{-s/2} L^p(\mathbb{R}^n), \quad \| \cdot \|_{s,p} \equiv \| (1 - \Delta)^{s/2} \|_p, \]

with $L^p_2(\mathbb{R}^n) = H^s(\mathbb{R}^n)$

(b) the pair of indices $(q, p)$ in (2.1) are given by the Strichartz estimates (see [60] and [21]):

\[ (\int_{-\infty}^{\infty} \| e^{it\Delta} u_0 \|_p^q dt)^{1/q} \leq c \| u_0 \|_2, \]

where

\[ \frac{n}{2} = \frac{2}{q} + \frac{n}{p}, \quad 2 \leq p \leq \infty, \text{ if } n = 1, \quad 2 \leq p < 2n/(n - 2), \text{ if } n \geq 2. \]

The value $s_c = n/2 - 2/(a - 1)$ in Theorem A is determined by a scaling argument: if $u(x, t)$ is a solution of the IVP associated to the NLS equation (1.1), then $u_\lambda(x, t) = \lambda^{2/(a-1)} u(\lambda x, \lambda^2 t)$ satisfies the same equation with data $u_\lambda(x, 0) = \lambda^{2/(a-1)} u_0(\lambda x)$. Hence, for $s \in \mathbb{R}$

\[ \| D^s u_\lambda(x, 0) \|_2 = c \| \xi^{s/2} \hat{u}(\xi, 0) \|_2 \]

is independent of $\lambda$ when $s = s_c$. In Theorem A the case (I) corresponds to the sub-critical case and (II) to the critical one. In the latter, one has that if $\| D^{s_c} u_0 \|_2$ is sufficiently small, then the local solution extends globally in time.

For the optimality of the results in Theorem A see [4], [11], and [40].

Formally, solutions of the NLS equation (1.1) satisfies the following conservation laws:

\[ \| u(\cdot, t) \|_2 = \| u_0 \|_2, \]

and

\[ E(t) = \int_{\mathbb{R}^n} (|\nabla_x u(x, t)|^2 + \frac{2\mu}{a+1} |u(x, t)|^{a+1}) dx = E(0). \]

Using these conservation laws one can extend the LWP results in Theorem A to a GWP one, for details we refer to [6], [64], and references therein.
Concerning the persistence properties in weighted Sobolev spaces of solutions of the IVP associated to the NLS equation (1.1) one has the following result established in [26], [27], and [28].

**Theorem B.** In addition to the hypothesis in Theorem A assume \( u_0 \in L^2(|x|^{2m}dx) \), \( m \in \mathbb{Z}^+ \) with \( m \leq a-1 \) if \( a \) is not an odd integer.

(I) If \( s \geq m \), then

\[
(2.5) \quad u \in C([-T,T] : H^s \cap L^2(|x|^{2m}dx)) \cap L^q([-T,T] : L^p(|x|^{2m}dx) = Z_T^{s,m}.
\]

(II) If \( 1 \leq s < m \), then (2.5) holds with \([s]\) instead of \( m\) and

\[
(2.6) \quad \Gamma^\beta u = (x_j + 2it\partial_{x_j})^\beta u \in C([-T,T] : L^2) \cap L^q([-T,T] : L^p),
\]

for any \( \beta \in (\mathbb{Z}^+)^n \) with \(|\beta| \leq m\).

The proof of Theorem B (see [26], [27], [28]) combines the operators ("vector fields")

\[
(2.7) \quad \Gamma_j = x_j + 2it\partial_{x_j} = e^{i|x|^2/4t}2it\partial_{x_j}(e^{-i|x|^2/4t}) = e^{it\Delta}x_j e^{-it\Delta}, \quad j = 1, \ldots, n,
\]

their commutative relation

\[
(2.8) \quad (i\partial_t + \Delta)\Gamma_j u = \Gamma_j(i\partial_t u + \Delta u), \quad j = 1, \ldots, n,
\]

so that \( e^{it\Delta}(x_j u_0) = \Gamma_j e^{it\Delta}u_0 \), and the structure of the nonlinearity in (1.1).

It should be remarked that Theorem B shows that the amount of decay in \( L^2(|x|^{2m}dx) \) preserved by the solution depends on the regularity in the Sobolev scale \( H^s \), \( s \geq 0 \) of the data, and the non-preserved decay is transformed in "local regularity". In particular, (2.6) tells us that \( t^\beta \partial^\beta_x u \in L^2_{loc}(\mathbb{R}^n) \), for \(|\beta| \leq m\) and \( t \in [-T,T] - \{0\} \).

Also one notices that the power of the weight \( m \) in Theorem B is assumed to be an integer. In [53] we were able to remove this restriction.

**Theorem 1.** In addition to the hypothesis in Theorem A assume \( u_0 \in L^2(|x|^{2m}dx) \), \( m > 0 \) with \([m] \leq a-1 \) if \( a \) is not an odd integer.

(I) If \( s \geq m \),

\[
(2.9) \quad u \in C([-T,T] : H^s \cap L^2(|x|^{2m}dx)) \cap L^q([-T,T] : L^p(|x|^{2m}dx) = Z_T^{s,m}.
\]

(II) If \( 1 \leq s < m \), then (2.9) holds with \([s]\) instead of \( m\) and

\[
(2.10) \quad \Gamma^\beta \Gamma^\gamma u(\cdot,t) \in C([-T,T] : L^2) \cap L^q([-T,T] : L^p),
\]

where \( \Gamma^\beta = e^{i|x|^2/4t}2it\partial_{x_j} \left( e^{-i|x|^2/4t} \right) \) with \(|\beta| = [m]\) and \( b = m - [m]\).

In particular,

\[
(2.11) \quad t^m \partial^\beta_x D^b u(\cdot,t) \in L^2_{loc}(\mathbb{R}^n), \quad |\beta| = [m], \quad b = m - [m], \quad t \in (-T,T) - \{0\}.
\]

As an application of this result we also prove that the persistence property in these weighted spaces can only hold for regular enough solutions. More precisely:

**Lemma 1.** Let \( u \) be a solution of the IVP associated to the NLS equation (1.1) provided by Theorem A. If there exist two times \( t_1, t_2 \in [0,T] \), \( t_1 \neq t_2 \) such that

\[
(2.12) \quad |x|^m u(t_1), \quad |x|^m u(t_2) \in L^2(\mathbb{R}^n), \quad m > s,
\]
m \leq a - 1 \text{ if } a \text{ is not an odd integer, then}

u \in C([-T, T] : H^m \cap L^2(|x|^{2m}dx)) \cap L^2([-T, T] : L^p_m \cap L^p(|x|^{2m}dx)).

Moreover, if \( a \) is an odd integer and (2.12) holds for all \( m \in \mathbb{Z}^+ \), then

\begin{equation}
\tag{2.13}
u \in C([-T, T] : S(\mathbb{R}^n)).
\end{equation}

A key ingredient in our proof was an appropriate version of the Leibnitz rule for homogeneous fractional derivatives of order \( b \in \mathbb{R} \)

\begin{equation}
\tag{2.14}
D^b f(x) \equiv ((2\pi|\xi|)^b \hat{f})(x)
\end{equation}
deduced as a direct consequence of the characterization of the \( L^p_\chi(\mathbb{R}^n) \) spaces (see (2.2) given in [58].

**Theorem D.** Let \( b \in (0, 1) \) and \( 2n/(n + 2b) \leq p < \infty \). Then \( f \in L^p_\chi(\mathbb{R}^n) \) if and only if

\begin{enumerate}
\item[(a)] \( f \in L^p(\mathbb{R}^n) \),
\item[(b)] \( D^b f(x) = \left( \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2b}} \, dy \right)^{1/2} \in L^p(\mathbb{R}^n) \),
\end{enumerate}

with

\begin{equation}
\tag{2.16}
\|f\|_{b,p} = \|(1 - \Delta)^{b/2} f\|_p \simeq \|f\|_p + \|D^b f\|_p \simeq \|f\|_p + \|D^b f\|_p.
\end{equation}

For the proof of Theorem D we refer to [58], where the optimality of the lower bound \( 2n/(n + 2b) \) was also established. The case \( p = 2n/(n + 2b) \) was proven in [18]. For a detailed discussion on the different characterizations of the \( L^p_\chi(\mathbb{R}^n) \) spaces we refer to [58] and [59].

It is easy to see that for \( p = 2 \) and \( b \in (0, 1) \) one has

\begin{equation}
\tag{2.17}
\|D^b f\|_2 \simeq \|D^b f\|_2,
\end{equation}

\begin{equation}
\tag{2.18}
\|D^b(fg)\|_2 \leq c(\|f\|_\chi \chi^b g + \|g\|_\chi \chi^b f),
\end{equation}

and for \( p > 2n/(n + 2b) \)

\begin{equation}
\tag{2.19}
\|D^b(fg)(x) \leq \|f\|_\infty \chi^b g(x) + \|g(x)\|_\chi \chi^b f(x).
\end{equation}

We observe that in (2.18) both terms on the right hand side are estimates on the product of functions. We do not know whether or not (2.18) still holds with \( D^b \) instead of \( D^b \) or for \( p \neq 2 \).

Theorem D (i.e. the estimates (2.18)-(2.17)) allows us to get the following inequalities:

(i) Let \( b \in (0, 1) \). For any \( t > 0 \)

\begin{equation}
\tag{2.20}
\chi^b e^{it|x|^2} \leq c(t^{b/2} + t^b |x|^b).
\end{equation}

(ii) Let \( b \in (0, 1) \). Then there exists \( c = c(b) > 0 \) such that for any \( t \in \mathbb{R} \)

\begin{equation}
\tag{2.21}
|||x|^b e^{it\Delta} f||_2 \leq c(t^{b/2} \|f\|_2 + t^b \|D^b f\|_2 + |||x|^b f||_2).
\end{equation}

(iii) Defining the operator \( I^b \) for \( b > 0 \) as in Theorem 1 (see (2.10))

\begin{equation}
\tag{2.22}
I^b \equiv I^b(t) = e^{it|x|^2/4t^2} \chi^b D^b \left( e^{-i|x|^2/4t^2} \right),
\end{equation}
one has for $b > 0$ and $t \in \mathbb{R}$ that
\begin{equation}
(2.23) 
\Gamma^b(t)e^{it\Delta}f = e^{it\Delta(|x|^b f)},
\end{equation}
and consequently
\begin{equation}
(2.24) 
\Gamma^b(t)f = e^{it\Delta(|x|^b e^{-it\Delta} f)}.
\end{equation}

In addition to the estimates (2.20)-(2.24) the following two lemmas were essential in the proof of Theorem 1 given in [53]. The first is a version of the Gagliardo-Nirenberg inequality for fractional derivatives.

**Lemma 2.** Let $1 < q, p, r < \infty$ and $0 < \alpha < \beta$. Then
\begin{equation}
(2.25) 
\|D^\alpha f\|_p \leq c\|f\|_1^{1-\theta}\|D^\beta f\|_q^\theta,
\end{equation}
with
\begin{equation}
(2.26) 
\frac{1}{p} - \frac{\alpha}{n} = (1-\theta)\frac{1}{r} + \theta\left(\frac{1}{q} - \frac{\beta}{n}\right), \quad \theta \in [\alpha/\beta, 1].
\end{equation}

The second is an interpolation estimate, which as Lemma 2, is a consequence of the three line theorem.

**Lemma 3.** Let $a, b > 0$. Assume that $J^a f = (1-\Delta)^{a/2} f \in L^2(\mathbb{R})$ and $\langle x \rangle^b f = (1+|x|^2)^{b/2} f \in L^2(\mathbb{R})$. Then for any $\theta \in (0, 1)$
\begin{equation}
(2.27) 
\|J^{\theta a}(\langle x \rangle^{(1-\theta)b} f)\|_2 \leq c\|\langle x \rangle^b f\|_2^{1-\theta}\|J^a f\|_2^\theta.
\end{equation}

For the study of persistence properties of the solution to the IVP associated to the NLS equation (1.1) in exponential weighted spaces we refer to [16], [17], and references therein.

Next, we shall consider the $k$-gKdV equation (1.2).

### 3. The $k$-generalized Korteweg-de Vries equation (1.2)

The following theorem describes the LWP theory in the classical Sobolev spaces $H^s(\mathbb{R})$ for the IVP associated to the $kgKdV$ equation (1.2).

**Theorem E.**

(I) The IVP associated to the equation (1.2) with $k = 1$ is LWP in $H^s(\mathbb{R})$ for $s \geq s_1^* = -3/4$.

(II) The IVP associated to the equation (1.2) with $k = 2$ is LWP in $H^s(\mathbb{R})$ for $s \geq s_2^* = 1/4$.

(III) The IVP associated to the equation (1.2) with $k = 3$ is LWP in $H^s(\mathbb{R})$ for $s \geq s_3^* = -1/6$.

(IV) The IVP associated to the equation (1.2) with $k \geq 4$ is LWP in $H^s(\mathbb{R})$ for $s \geq s_k^* = (k-4)/2k$.

The result $s > -3/4$ for the case $k = 1$ was established in [39]. The limiting value $s = -3/4$ was obtained in [11], [24], and [42]. The result for the case $k = 2$ was proven in [38]. The result $s > -1/6$ for the case $k = 3$ was given in [22]. The limiting value $s = -1/6$ was obtained in [63]. The proof of the cases $k \geq 4$ was given in [38].

The above local results apply to both real and complex valued functions.

The scaling argument described in (2.4) affirms that LWP should hold for $s \geq s_k = (k-4)/2k$. As Theorem E shows this is the case for $k \geq 3$ (where for $s_k = s_k^*$
one has \( T = T(u_0) \). However, in the cases \( k = 1 \) and \( k = 2 \) the values suggested by the scaling do not seem to be reachable in the Sobolev scale, see [40], and [11]. For the sharpness of these results we refer to [4], [40], and [11].

Real valued solutions of the \( k \)-gKdV equation (1.2) formally satisfy at least three conservation laws:

\[
I_1(u) = \int_{-\infty}^{\infty} u(x,t) dx, \quad I_2(u) = \int_{-\infty}^{\infty} (u(x,t))^2 dx, \\
I_3(u) = \int_{-\infty}^{\infty} ((\partial_x u(x,t))^2 - \frac{2}{(k+1)(k+2)} u(x,t)^{k+2}) dx.
\]

It was proven in [13] that for \( k = 1 \) and \( k = 2 \) one has global well posedness for \( s > -3/4 \) and \( s > 1/4 \), respectively. The global cases for \( k = 1, s = -3/4 \) and \( k = 2, s = 1/4 \) were proven in [24] and [42]. For the case \( k = 3 \) the global well posedness is known for \( s > -1/42 \), see [23].

For \( k = 4 \) blow up of “large” enough solutions was proven in [48]. Similar results for \( k \geq 5 \) remain an open problem.

Concerning the persistence of these solutions in weighted Sobolev spaces one has the following result found in [34].

**Theorem F.** Let \( m \in \mathbb{Z}^+ \). Let \( u \in C([-T,T] : H^s(\mathbb{R})) \cap \ldots \) with \( s \geq 2m \) be the solution of the IVP associated to the equation (1.2) provided by Theorem E. If \( u(x,0) = u_0(x) \in L^2(|x|^{2m}dx) \), then

\[
u \in C([-T,T] : H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)).
\]

We recall that if for a solution \( u \in C([0,T] : H^s(\mathbb{R})) \) of (1.2) one has that \( \exists t_0 \in [0,T] \) such that \( u(\cdot, t_0) \in H^{s'}(\mathbb{R}), s' > s \), then \( u \in C([0,T] : H^{s'}(\mathbb{R})) \). So we shall mainly consider the most interesting case \( s = 2m \) in Theorem F.

The proof of Theorem F combines the operator

\[
\Gamma = x + 3t \partial_x^2,
\]

and its commutative relation with the linear part \( L = \partial_t + \partial_x^4 \) of the equation (1.2) i.e.

\[
\Gamma(\partial_t + \partial_x^4)v = (\partial_t + \partial_x^4)\Gamma v.
\]

As in the case of the NLS equation (1.1) we would like to extend Theorem F where \( m \in \mathbb{Z}^+ \) to the case \( m \in \mathbb{R}, m > 0 \). Our first result in this direction is the following:

**Theorem 2.** Let \( m \geq 0 \). Let \( u \in C([-T,T] : H^m(\mathbb{R})) \cap \ldots \) with \( m \geq \max\{s_k^*; 0\} \) be the solution of the IVP associated to the equation (1.2) provided by Theorem E. If \( u(x,0) = u_0(x) \in L^2(|x|^m dx) \), then

(I) If \( m < 1 \), then for any \( \epsilon > 0 \)

\[
u \in C([-T,T] : H^m(\mathbb{R}) \cap L^2(|x|^{m-\epsilon} dx)).
\]

(II) If \( m \geq 1 \), then

\[
u \in C([-T,T] : H^m(\mathbb{R}) \cap L^2(|x|^m dx)).
\]

In [51] and [52] the loss of power \( \epsilon > 0 \) in the weight when \( m < 1 \) was removed for the equation (1.2) with non-linearity \( k = 2,4,5, \ldots \). More precisely, the following optimal result was established in [52]:

\[
\int_{-\infty}^{\infty} (u(x,t))^2 dx = \int_{-\infty}^{\infty} ((\partial_x u(x,t))^2 - \frac{2}{(k+1)(k+2)} u(x,t)^{k+2}) dx.
\]
Theorem 3. Let $m \geq \max\{s_k^*; 0\}$ with $k = 2, 4, 5, \ldots$. Let $u \in C([-T, T] : H^m(\mathbb{R})) \cap \ldots$ be the solution of the IVP associated to the equation (1.2) provided by Theorem E. If $u(x, 0) = u_0(x) \in L^2(|x|^m dx)$, then

$$u \in C([-T, T] : H^m(\mathbb{R}) \cap L^2(|x|^m dx)).$$

It should be remarked that in the cases $k = 1$ and $k = 3$ the proof of the local theory in Theorem E is based on the spaces $X_{s, b}$ introduced in the context of dispersive equations in [5]. For all the other powers $k$ one has a local existence theory based on a contraction principle in a space defined by mixed norms of the type $L^p(\mathbb{R} : L^q([0, T]))$ or $L^q([0, T] : L^p(\mathbb{R}))$ (see [38]). This is the main difficulty in extending the optimal result in Theorem 3 to the powers $k = 1$ and $k = 3$ in (1.2).

Proof of Theorem 2

We shall sketch the ideas in the proof of Theorem 2 and refer to [51] and [52] for the justification of the argument and further details.

Following Kato’s idea in [34] to establish the local smoothing effect (i.e. multiplying the equation (1.2) by $u(x, t)\phi(x)$, integrating the result, and using integration by parts) one formally gets the identity

$$\frac{d}{dt} \int u^2 \phi dx + 3 \int (\partial_x u)^2 \phi' dx - \int u^2 \phi'' dx - \frac{2}{k+2} \int u^{k+2} \phi' dx = 0. \tag{3.1}$$

Let us consider first the case $max\{s_k^*; 0\} \leq m < 1$. From the local theory one has the following estimates for the solution $u = u(x, t)$

$$\sup_{x \in \mathbb{R}} (\int_0^T \int (\partial_x D^m_x u(x, t))^2 dt) < c_T \| J^m u_0 \|_2 = c_T \| u_0 \|_{m, 2}, \tag{3.2}$$

the sharp form of the local smoothing effect found in [37]-[38]), and

$$\| D^m_x u \|_{L^2 \tilde{L}_T^2} = \left( \int_{-\infty}^\infty \int_0^T \| D^m_x u(x, t) \|^2 dt dx \right)^{1/2} \leq T^{1/2} \sup_{t \in [0, T]} \| D^m_x u(t) \|_2 < c_T \| D^m u_0 \|_2 \leq c_T \| u_0 \|_{m, 2}. \tag{3.3}$$

Now, we consider the extensions of the estimates in (3.2)-(3.3) to the operators $D_x^{1+m+iy}$ and $D_x^{m+iy}$, $y \in \mathbb{R}$ respectively. First, in the linear case one has the estimates

$$\| D_x^{m+1+iy} v \|_{L^\infty \tilde{L}_T^2} \leq c_T \| D^m v_0 \|_2, \tag{3.4}$$

for

$$v(x, t) = U(t)v_0(x) = c \int_{-\infty}^\infty e^{ix\xi} e^{ity} \hat{v}_0(\xi) d\xi. \tag{3.5}$$

To apply the three line theorem we consider the function $F(z)$ defined on $S = \{ z \in \mathbb{C} : \Re(z) \in [0, 1] \}$

$$F(z) = \int_{-\infty}^\infty \int_0^T D^m_x(z) v(x, t) \phi(x, z) f(t) dt dx,$$
where
\[ s(z) = (1-z)(1+m) + zm, \quad 1/q(z) = (1-z) + z/2, \quad q = 2/(2-m), \]
\[ \phi(x, z) = |g(x)|^{q(z)} \frac{g(x)}{|g(x)|}, \quad \text{with} \quad \|g\|_{L^2(z^{2-m})} = \|f\|_{L^2([0, T])} = 1, \]
which is analytic on the interior of \( S \). So using that
\[ \|\phi(r, 0 + iy)\|_1 = \|\phi(r, 1 + iy)\|_2 = 1, \]
one gets that
\[ \|\partial_x v\|_{L^{2/m}_x L^{2}_t} \leq c\|D_x v\|_{L^{2/m}_x L^{2}_t} \]
(3.6)
\[ \leq c \sup_{y \in \mathbb{R}}\|D_x^{1+m+iy} v\|_{L^{2/m}_x L^{2}_t}^{1-m} \sup_{y \in \mathbb{R}}\|D_x^{m+iy} v\|_{L^{2}_x L^{2}_t}^m \leq cT\|D^m v_0\|_2. \]
Inserting the estimate (3.6) in the proof of the local well posedness one obtains that
\[ \|\partial_x u\|_{L^{2/m}_x L^{2}_t} \leq cT\|u_0\|_{m, 2}, \]
for \( u = u(x, t) \) solution of the \( k\)-gKdV equation (1.2).

Now taking \( \phi(x) = \langle x \rangle^{m-\epsilon}, \quad \epsilon > 0 \) sufficiently small in (3.1), (we recall that \( m < 1 \)) and integrating in the time interval \([0, T]\) one finds that
\[ \int_0^T \int_{-\infty}^{\infty} (\partial_x u(x, t))^2 \phi(x) dx dt = c\|\partial_x u(x) \|^2 \frac{1}{2} - \frac{5}{2} \|\partial_x^2 v\|_{L^2_x L^2_t} \]
(3.8)
\[ \leq c\|\langle x \rangle^{m/2-1/2-\epsilon/2}\|_{L^{2/m}_x L^{2}_t} \|\partial_x u\|_{L^{2/m}_x L^{2}_t} \leq c_{m, \epsilon}\|\partial_x u\|_{L^{2/m}_x L^{2}_t}, \]
which combined with (3.6) and (3.1) shows that \( \langle x \rangle^{m/2-\epsilon/2} u(\cdot, t) \in L^2(\mathbb{R}) \) for \( t \in [0, T] \). This basically completes the proof of the case \( m < 1 \).

Next, we shall consider the case \( m \geq 1 \).

We take in (3.1) \( \phi(x) = \langle x \rangle^m \) in (3.1), so we need to estimate the term
\[ \int_{-\infty}^{\infty} |\partial_x u(x, t)|^2 \langle x \rangle^{m-1} dx = \|\partial_x u(\cdot, t)\langle \cdot \rangle^{m-1/2}\|_{L^2_x}^2. \]
Thus, combining Lemma 3 in the previous section, the preservation of the \( L^2 \)-norm of the solution, and Lemma 3 it follows that
\[ \|\partial_x u(\cdot, t)\langle \cdot \rangle^{m-1/2}\|_2 \]
(3.9)
\[ \leq \|\partial_x J^{-1} J(u(\cdot, t)\langle \cdot \rangle^{m-1/2})\|_2 + c\|u(\cdot, t)\langle \cdot \rangle^{m-3/2}\|_2 \]
\[ \leq \|\partial_x J^{-1} J(u(\cdot, t)\langle \cdot \rangle^{m-1/2})\|_2 + c\|u(\cdot, t)\langle \cdot \rangle^{m/2}\|_2 \]
\[ \leq c\|J^m u(\cdot, t)\|_2^{1/m} \|\langle x \rangle^{m/2} u(\cdot, t)\langle \cdot \rangle^{m-1/2}\|_2^{1-1/m} + c\|u(\cdot, t)\langle \cdot \rangle^{m/2}\|_2. \]
Hence, inserting (3.9) in (3.1), using Young and Gronwall inequalities, the hypothesis \( m \geq 1 \), and the fact that the \( H^m \)-norm of the solution is bounded in the time interval \([0, T]\) one obtains the desired result
\[ \sup_{t \in [0, T]} \|\langle x \rangle^{m/2} u(\cdot, t)\|_{L^2} < \infty. \]
This completes the sketch of the proof of Theorem 2.
To finish this section concerning the $k$-gKdV equation (1.2) we will make some comments concerning the proof of Theorem 3 given in [51] and [52]. One of the key elements in that proof is the following commutator estimate:

**Lemma 4.** Let $0 < \alpha < 1$ and $1 < p < \infty$. Then for functions $f, g : \mathbb{R} \to \mathbb{C}$ one has that

$$
\|D^{\alpha}(fg) - fD^{\alpha}g\|_p \leq c\|Q_N(D^{\alpha}f)\|_{L^{\infty}}\|g\|_2,
$$

where

$$
\|Q_N(f)\|_{L^{\infty}} \equiv \left\| \sum_{N \in \mathbb{Z}} |Q_N(f)| \right\|_{L^{\infty}},
$$

and

$$
Q_N(f)(x) = \left( (\eta \left( \frac{x}{2^N} \right) + \eta \left( -\frac{x}{2^N} \right) ) \right)^{\gamma}(x),
$$

where $\eta \in C_0^\infty(\mathbb{R})$ with $\text{supp}(\eta) \subseteq [1, 2, 2]$ so that

$$
\sum_{N \in \mathbb{Z}} \left( \eta \left( \frac{x}{2^N} \right) + \eta \left( -\frac{x}{2^N} \right) \right) = 1, \quad \text{for } x \neq 0.
$$

In the proof of Theorem 3 for the case $k = 2$ and $m = 1/4$ (extrimal case) given in [51] Lemma 4 was combined with the inequality

$$
\|D^{1/8}\xi Q_N \left( \frac{e^{i}\xi^3}{(1+\xi^2)^{1/8}} \right) \|_{L^{\infty}} < \infty,
$$

to establish the main estimate in the proof.

For the study of persistence properties of the solution to the IVP associated to the $k$-gKdV equation (1.2) in exponential weighted spaces we refer to [41] and [15] and references therein.

Finally, we shall consider the BO equation (1.3).

**4. THE BENJAMIN-ONO EQUATION (1.3)**

The LWP in the Sobolev spaces $H^s(\mathbb{R})$ of the IVP associated to the BO equation (1.3) has been largely considered: in [1] and [32] LWP was established for $s > 3/2$, in [56] for $s \geq 3/2$, in [44] for $s > 5/4$, in [36] for $s > 9/8$, in [62] for $s \geq 1$, in [7] for $s > 1$, and in [31] LWP was proven in $H^s(\mathbb{R})$ for $s \geq 0$.

Real valued solutions of the IVP (1.3) satisfy infinitely many conservation laws (time invariant quantities), the first three are the following:

$$
\begin{align*}
I_1(u) &= \int_{-\infty}^{\infty} u(x,t)dx, \\
I_2(u) &= \int_{-\infty}^{\infty} u^2(x,t)dx, \\
I_3(u) &= \int_{-\infty}^{\infty} (|D^{1/2}_x u|^2 - \frac{u^3}{3})(x,t)dx,
\end{align*}
$$

(4.1)

where $D_x = \mathcal{H} \partial_x$.

The $k$-conservation law $I_k$ provides an *a priori* estimate of the $L^2$-norm of the derivatives of order $(k - 2)/2$, $k > 2$ of the solution, i.e. $\|D^{(k-2)/2}_x u(t)\|_2$. This allows one to deduce GWP from LWP results.
In the BO equation the dispersive effect is described by a non-local operator and is significantly weaker than that exhibited by the Korteweg-de Vries (KdV) equation, i.e. \( k = 1 \) in (1.2). Indeed, it was proven in [49] that for any \( s \in \mathbb{R} \) the map data-solution from \( H^s(\mathbb{R}) \) to \( C([0, T] : H^s(\mathbb{R})) \) is not locally \( C^2 \), and in [45] that it is not locally uniformly continuous. In particular, this implies that no LWP results can be obtained by an argument based only on a contraction method.

Consider the weighted Sobolev spaces

\[
(4.2) \quad Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \quad \text{and} \quad \dot{Z}_{s,r} = \{ f \in Z_{s,r} : \hat{f}(0) = 0 \} \quad s, r \in \mathbb{R}.
\]

In [32] the following results were obtained:

**Theorem G.**

(I) The IVP associated to the BO equation (1.3) is GWP in \( Z_{2,2} \).

(II) If \( \tilde{u}_0(0) = 0 \), then the IVP associated to the BO equation (1.3) is GWP in \( \dot{Z}_{3,3} \).

(III) If \( u(x, t) \) is a solution of the IVP associated to the BO equation (1.3) such that \( u \in C([0, T] : Z_{4,4}) \) for arbitrary \( T > 0 \), then \( u(x, t) \equiv 0 \).

We observe that the linear part of the equation in (1.3) \( L = \partial_t + \mathcal{H}\partial_x^2 \) commutes with the operator \( \Gamma = x - 2t\mathcal{R}_x \), i.e.

\[
[L, \Gamma] = L\Gamma - \Gamma L = 0.
\]

Also, the solution \( v(x, t) \) of the associated IVP

\[
(4.3) \quad v(x, t) = U(t)v_0(x) = e^{-it\mathcal{H}\partial_x^2}v_0(x) = (e^{-it\xi|\xi|^\gamma}v_0)^\gamma(x),
\]

satisfies that \( v(\cdot, t) \in L^2(|x|^{2k}dx), t \in [0, T], \) when \( v_0 \in Z_{k,k}, k \in \mathbb{Z}^+ \) for \( k = 1, 2, \ldots \) and

\[
\int_{-\infty}^\infty x^j v_0(x)dx = 0, \quad j = 0, 1, \ldots, k - 3, \quad \text{if} \quad k \geq 3.
\]

In [33] the unique continuation result in \( Z_{4,4} \) in Theorem G was improved:

**Theorem I.** Let \( u \in C([0, T] : H^2(\mathbb{R})) \) be a solution of the IVP (1.3). If there exist three different times \( t_1, t_2, t_3 \in [0, T] \) such that

\[
(4.4) \quad u(\cdot, t_j) \in Z_{4,4}, \quad j = 1, 2, 3, \quad \text{then} \quad u(x, t) \equiv 0.
\]

As in the previous cases, the goal was to extend the results in Theorem G and Theorem I from integer values to the continuum optimal range of indices \((s, r)\). In this direction one finds the following results established in [19]:

**Theorem 4.**

(I) Let \( s \geq 1, r \in [0, s], \) and \( r < 5/2. \) If \( u_0 \in Z_{s,r}, \) then the solution \( u(x, t) \) of the IVP associated to the BO equation (1.3) satisfies that \( u \in C([0, \infty) : Z_{s,r}). \)

(II) For \( s > 9/8 (s \geq 3/2), \) \( r \in [0, s], \) and \( r < 5/2 \) the IVP associated to the BO equation (1.3) is LWP (GWP resp.) in \( Z_{s,r} \).

(III) If \( r \in [5/2, 7/2) \) and \( r \leq s, \) then the IVP associated to the BO equation (1.3) is GWP in \( \dot{Z}_{s,r} \).

**Theorem 5.** Let \( u \in C([0, T] : Z_{2,2}) \) be a solution of the IVP associated to the BO equation (1.3). If there exist two different times \( t_1, t_2 \in [0, T] \) such that

\[
(4.5) \quad u(\cdot, t_j) \in Z_{5/2,5/2}, \quad j = 1, 2, \quad \text{then} \quad \tilde{u}_0(0) = 0, \quad \text{(so} \ u(\cdot, t) \in \dot{Z}_{5/2,5/2}).
\]
Theorem 6. Let $u \in C([0, T] : \dot{Z}_{3,3})$ be a solution of the IVP (1.3). If there exist three different times $t_1, t_2, t_3 \in [0, T]$ such that
\begin{equation}
(u(\cdot, t_j) \in Z_{7/2, 7/2}, \ j = 1, 2, 3, \ \text{then} \ u(x, t) \equiv 0.
\end{equation}

We also refer readers to the related works [47], [25], and [43].

Remarks: Theorem 5 and Theorem 6 show that the upper values of $r$ for the persistence properties in $Z_{s,p}$ and $\dot{Z}_{s,k}$ in Theorem 4 are optimal. We recall that if $u \in C([0, T] : H^s(\mathbb{R}))$ is a solution of the BO equation (1.3) such that $\exists t_0 \in [0, T]$ for which $u(x, t_0) \in H^s(\mathbb{R})$, $s' > s$, then $u \in C([0, T] : H^s(\mathbb{R}))$. So it suffices to consider the most interesting case $s = r$ in (4.2).

The proof of Theorems 6 is based on weighted energy estimates and involves several inequalities concerning the Hilbert transform $H$.

Among them one finds the $A_p$ condition introduced in [50].

Definition 1. A non-negative function $w \in L^1_{loc}(\mathbb{R})$ satisfies the $A_p$ inequality with $1 < p < \infty$ if
\begin{equation}
\sup_{Q \text{ interval}} \left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} = c(w) < \infty,
\end{equation}
where $1/p + 1/p' = 1$.

It was proven in [30] that this is a necessary and sufficient condition for the Hilbert transform $H$ to be bounded in $L^p(w(x)dx)$ (see [30], i.e. $w \in A_p$, $1 < p < \infty$ if and only if
\begin{equation}
\left( \int_{-\infty}^{\infty} |Hf|^p w(x)dx \right)^{1/p} \leq c^* \left( \int_{-\infty}^{\infty} |f|^p w(x)dx \right)^{1/p},
\end{equation}
In the case $p = 2$, a previous characterization of $w$ in (4.7) was found in [29]. However, even though the main case is for $p = 2$, the characterization (4.7) will be the one used in the proof. In particular, one has that in $\mathbb{R}$
\begin{equation}
|x|^\alpha \in A_p \iff \alpha \in (-1, p-1).
\end{equation}

In order to justify some of the arguments in the proofs one need some further continuity properties of the Hilbert transform. More precisely, the proof requires the constant $c^*$ in (4.8) to depend only on $c(w)$ the constant describing the $A_p$ condition (see (4.7)) and on $p$. In [55] precise bounds for the constant $c^*$ in (4.7) were given which are sharp in the case $p = 2$ and sufficient for the purpose in [19].

It will be essential in the arguments in [19] that some commutator operators involving the Hilbert transform $H$ are of “order zero”. More precisely, one shall use the following estimate: $\forall p \in (1, \infty)$, $l, m \in \mathbb{Z}^+ \cup \{0\}$, $l+m \geq 1$ $\exists c = c(p; l; m) > 0$ such that
\begin{equation}
||\partial_x^l [H; a] \partial_x^m f||_p \leq c ||\partial_x^{l+m} a||_\infty ||f||_p.
\end{equation}
In the case $l + m = 1$, (4.10) is Calderón’s first commutator estimate [8]. The case $l + m \geq 2$ of the estimate (4.10) was proved in [14].

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