

Lower Bounds for Non-Trivial Traveling Wave Solutions of Equations of KdV Type

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Abstract

We prove that if a solution of an equation of KdV type is bounded above by a traveling wave with an amplitude that decays faster than a given linear exponential then it must be zero. We assume no restrictions neither on the size nor in the direction of the speed of the traveling wave.

1 Introduction and main results.

In this paper we will continue our study initiated in [2] on solutions of dispersive systems that decay like a linear exponential. In particular, we extend the results proved in that paper for traveling wave solutions of non-linear Schrödinger (NLS) equations to the case of equations of Korteweg–de Vries (KdV) type.

More concretely we shall consider solutions of equations of the form

$$(\partial_t + \partial_x^3) u = a(u) \partial_x u, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (1.1)$$

with a regular and such that

$$|a(s)| \leq M_1 (|s| + |s|^j), \quad j = 1, 2, 3, \dots \quad (1.2)$$

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Our main result is the following one.

Theorem 1 *Assume $u \in C^1(\mathbb{R} : H^1(\mathbb{R}))$ is a real solution of (1.1)–(1.2) with $a(u)$ also real. Then there exist $\lambda_0 \geq 1$ and $c_0 > 0$ such that if for some $\lambda > 0$, $b \in \mathbb{R}$*

$$\sup_{t \geq 0} \|u(\cdot, t)\|_{H^1}^2 \leq M_2, \quad (1.3)$$

$$\sup_{t \geq 0} \int e^{|x-bt|} |u(x, t)|^2 dx \leq M_3, \quad (1.4)$$

$$\sup_{t \geq 0} \int e^{2\lambda|x-bt|} |u(x, t)|^2 < +\infty, \quad (1.5)$$

and

$$\lambda \geq \max \left\{ \lambda_0, c_0 M_1^2 \left(1 + [M_3 + M_2^{1/2} M_3^{1/2}]^{j/2} \right) \right\},$$

then

$$u \equiv 0.$$

Relevant examples of non-linear potentials are the pure power ones

$$a(u) = -ju^j \quad j = 1, 2, 3, \dots, \quad (1.6)$$

and the completely integrable ones

$$a(u) = \alpha_1 u + \alpha_2 u^2, \quad (1.7)$$

that are usually known as Gardner equations, where the cases $(\alpha_1, \alpha_2) = (1, 0)$ and $(\alpha_1, \alpha_2) = (0, 1)$ are the KdV and modified KdV equation respectively.

Remark 1 *For (1.6) there is a scaling symmetry*

$$u_\lambda(x, t) = \lambda^{2/j} u(\lambda x, \lambda^3 t), \quad \lambda > 0,$$

and an explicit family of traveling waves

$$\lambda^{2/j} \varphi_j(\lambda(x - \lambda^2 t)), \quad \text{with } \varphi_j = \left(\frac{j+2}{2} \operatorname{sech}^2\left(\frac{j}{2}x\right) \right)^{\frac{1}{j}}.$$

Hence in this case we have traveling wave solutions that propagate to the right with velocity λ^2 .

Also given any j and using the scaling invariance we can always choose λ small enough such that there exists a non-trivial solution that verifies (1.3), (1.4), and (1.5).

Regarding negative velocities we recall the existence of breather solutions in the case of Gardner equations for some specific choices of the parameter α_1 and α_2 . Let us consider for simplicity the modified KdV equation; and specifically we take $a(u) = -6u^2$. Then, with this normalization the soliton solutions are

$$u(x, t) = \lambda \operatorname{sech}(\lambda(x - \lambda^2 t)), \quad (1.8)$$

and the breather solutions are [11]

$$u(x, t) = 2\lambda \operatorname{sech}(\lambda(x + \gamma t)) \cdot \left\{ \frac{\cos[\Phi(x, t)] - \frac{\lambda}{\mu} \sin \Phi(x, t) \tanh(\lambda(x + \gamma t))}{1 + \left(\frac{\lambda}{\mu}\right)^2 \sin^2(\Phi(x, t)) \operatorname{sech}^2(\lambda(x + \gamma t))} \right\}, \quad (1.9)$$

with

$$\gamma = 3\mu^2 - \lambda^2, \quad \delta = \mu^2 - 3\lambda^2, \quad \text{and} \quad \Phi = \mu(x + \delta t) - \tan^{-1}\left(\frac{\lambda}{\mu}\right).$$

These breathers can be seen as wave packets with an amplitude and a frequency determined by the parameters λ and μ . The velocity of the amplitude is $\gamma = 3\mu^2 - \lambda^2$ and it can take any real value.

Remark 2 Notice that in the above examples the decay rate of the amplitude is just given by λ as in the statement of our theorem. Choosing as before λ small enough we also conclude the sharpness of our result for negative velocities.

Remark 3 Breather solutions also exist for complex KdV equations. For example if we choose $a(u) = -|u|^2$ with $u(x, t) \in \mathbb{C}$ it is easy to check (see for example [7]) that

$$u(x, t) = \sqrt{3}e^{-it\mu(3\lambda^2 - \mu^2) + i\mu x} \psi_\lambda(x - (\lambda^2 - 3\mu^2)t),$$

with $\psi_\lambda(x) = \lambda\psi_1(\lambda x)$ and $\psi_1(x) = \sqrt{2}\operatorname{sech}x$. The arguments needed for the proof of Theorem 1 also work for this complex KdV (i.e. $a(u) = -|u|^2$).

There are three fundamental ingredients in the proof of Theorem 1. The first one is that the L^2 norm is a conserved quantity. This property is needed to conclude that $u(x, 0) = 0$ if for some sequence of times t_n we have that

$$\lim_{t_n \rightarrow \infty} \|u(\cdot, t_n)\|_{L^2} = 0.$$

The second ingredient is Kato's theory in [5] concerning the persistent property of solutions to the initial value problem with data that satisfies linear exponential decay. Kato's result is used to prove that if (1.5) holds then a similar inequality is also true for $\partial_x^k u$ $k = 1, 2, 3$. We require this property since instead of working with the equation (1.1)–(1.2) we use the one satisfied by $f = e^{\lambda\theta}u$ with $\theta(x, t)$ an appropriate Carleman weight that grows almost linearly at infinity.

The final ingredient is the convexity of $H(t) = \|f\|_{L^2}^2 = \langle f, f \rangle$. As done in [2] and [3] we estimate $\dot{H}(t)$ and $\ddot{H}(t)$ using integration by parts, therefore we need f and its spatial derivatives in L^2 . At this point we closely follow the arguments in [2] but the algebra, as it can be expected, turns out to be more complicated in the KdV setting than in the NLS one. The details can be found in the proof of Claim 1 in Section 2. In this claim we establish the positivity of the commutator that appears in the computation of $\ddot{H}(t)$ -see also [9] and [10].

There is still another difference with respect to [2]. For generalized KdV (1.1)–(1.2) the non-linear potential can not be treated in a perturbation way and some structure is needed (see the proof of Claim 2 in Section 2). This structure still holds in the complex KdV equation that we mentioned above.

The use of Carleman weights to obtain positive commutators is a standard technique in elliptic theory for obtaining lower bounds of eigenfunctions of a Schrödinger operator -see [1], [4], [8]. In the time evolution setting some modifications of this technique are needed and an important step in our approach is the use of the two identities (2.5) and (2.6). These identities also appear in [2].

The rest of the paper, that is to say Section 2, is devoted to the proof of Theorem 1.

2 Proof of Theorem 1.

From the results by T. Kato in [5] we can assume that (1.5) is also satisfied possibly with another λ for $\partial_x^k u(x, t)$ with $k = 1, 2, 3$.

As in [2] we shall work with $f = e^{\lambda\theta} u$ with $\theta = \theta(x, t)$. More concretely $\theta(x, t) = \varphi(r)$, $r = |x - bt|$ for some regular even φ that will be fixed later on and that it grows at most linearly at infinity. As in [3] we have

$$\begin{aligned}
e^{\lambda\theta} (\partial_t + \partial_x^3) e^{-\lambda\theta} f &= (S_\lambda + A_\lambda) f; \\
S_\lambda &= -3\lambda \partial_x (\partial_x \theta \partial_x) + (-\lambda^3 (\partial_x \theta)^3 - \lambda \partial_x^3 \theta - \lambda \partial_t \theta); \\
A_\lambda &= \partial_t + \partial_x^3 + 3\lambda^2 (\partial_x \theta)^2 \partial_x + 3\lambda^2 \partial_x \theta \partial_x^2 \theta = \partial_t + \tilde{A}_\lambda; \\
A_\lambda^* &= -A_\lambda \quad \tilde{A}_\lambda^* = -\tilde{A}_\lambda \quad S_\lambda^* = S_\lambda.
\end{aligned} \tag{2.1}$$

Hence, using the notation $\partial_x f = f_x$,

$$\begin{aligned}
\langle [S_\lambda; A_\lambda] f, f \rangle &= \langle (S_\lambda A_\lambda - A_\lambda S_\lambda) f, f \rangle \\
&= 9\lambda \int \varphi''(r) f_{xx}^2 + 6\lambda b \int \varphi''(r) f_x^2 + 18\lambda^3 \int (\varphi'(r))^2 \varphi''(r) f_x^2 \\
&\quad - 6\lambda \int \varphi^{IV}(r) f_x^2 + 9\lambda^5 \int (\varphi'(r))^4 \varphi''(r) f^2 + \lambda b^2 \int \varphi''(r) f^2 \\
&\quad - 3\lambda^3 \int (\varphi''(r))^3 f^2 - 18\lambda^3 \int \varphi'(r) \varphi''(r) \varphi'''(r) f^2 \\
&\quad - 3\lambda^3 \int (\varphi''(r))^2 \varphi^{IV}(r) f^2 + \lambda \int \varphi^{VI}(r) f^2 - 2\lambda b \int \varphi^{IV}(r) f^2 \\
&\quad - 6\lambda^3 b \int \varphi'(r)^2 \varphi''(r) f^2 \\
&= \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4} + \textcircled{5} + \textcircled{6} + \textcircled{7} + \textcircled{8} + \textcircled{9} + \textcircled{10} + \textcircled{11} + \textcircled{12}.
\end{aligned} \tag{2.2}$$

Notice that

$$\textcircled{5} + \textcircled{6} + \textcircled{12} = \lambda \int \left(3\lambda^2 (\varphi'(r))^2 - b \right)^2 \varphi''(r) f^2. \tag{2.3}$$

As in [2] we shall use two identities that hold for solutions of

$$\partial_t f = - \left(S_\lambda + \tilde{A}_\lambda \right) f + F, \tag{2.4}$$

with

$$\tilde{A}_\lambda^* = -\tilde{A}_\lambda \quad S_\lambda^* = S_\lambda.$$

First observe that from (2.4)

$$\frac{d}{dt} \langle f, f \rangle = -2 \langle S_\lambda f, f \rangle + 2 \langle F, f \rangle,$$

and, recalling that in our case $A_\lambda = \partial_t + \tilde{A}_\lambda$, we have

$$\frac{d}{dt} \langle S_\lambda f, f \rangle = -\langle (S_\lambda A_\lambda - A_\lambda S_\lambda) f, f \rangle - 2\langle S_\lambda f, S_\lambda f \rangle + 2\langle F, S_\lambda f \rangle.$$

In this last identity we have used that

$$S_\lambda A_\lambda - A_\lambda S_\lambda = S_\lambda \tilde{A}_\lambda - \tilde{A}_\lambda S_\lambda - (S_\lambda)_t.$$

Then, take $\eta : [a, b] \rightarrow \mathbb{R}$. A simple integration by parts gives (see Proposition 1 and Proposition 2 in [2] for more details)

$$\int_a^b \eta'(t) \langle S_\lambda f, f \rangle dt = -\frac{1}{2} (\eta' \langle f, f \rangle)|_a^b + \int_a^b \eta'(t) \langle F, f \rangle dt + \frac{1}{2} \int_a^b \eta''(t) \langle f, f \rangle dt, \quad (2.5)$$

and

$$\begin{aligned} \int_a^b \eta'(t) \langle S_\lambda f, f \rangle dt &= (\eta \langle S_\lambda f, f \rangle)|_a^b + \int_a^b \eta \langle (S_\lambda A_\lambda - A_\lambda S_\lambda) f, f \rangle dt \\ &+ 2 \int_a^b \eta \langle S_\lambda f, S_\lambda f \rangle dt - 2 \int_a^b \eta \langle F, S_\lambda f \rangle dt. \end{aligned} \quad (2.6)$$

Notice that if u solves (1.1)–(1.2) and $f = e^{\lambda\varphi(x-bt)}u$, then f solves

$$\partial_t f = -\left(S_\lambda + \tilde{A}_\lambda\right) f + a(u) (\partial_x f - \lambda\varphi'(x-bt)f). \quad (2.7)$$

Finally we shall use the following Carleman weight φ_0 :

$$\varphi_0 \in \mathcal{C}^6(\mathbb{R}), \text{ even and positive,} \quad (2.8)$$

$$\varphi_0'(r) = r \text{ if } 0 \leq r \leq 3/2 \text{ and } \varphi_0'(r) = 2 - \frac{\log 2}{4 \log r} \text{ if } r \geq 2, \quad (2.9)$$

$$0 < \varphi_0''(r) \leq 1 \text{ and it is a decreasing function for } r > 3/2, \quad (2.10)$$

there exists $c_0 > 0$ such that

$$\left| \frac{d^k}{dr^k} \varphi_0(r) \right| \leq c_0 \varphi_0''(r) \quad k = 3, 4, 5, 6. \quad (2.11)$$

The proof of the theorem will follow from the next three claims. Recall that $f = e^{\lambda\varphi(r)}u$ with $r = |x - bt|$. If $b \geq \frac{3}{2}\lambda^2$ we shall choose $\varphi = \frac{1}{4}\varphi_0$. Otherwise we shall take $\varphi = \varphi_0$. The reason behind these choices will become clear in the proof of Claim 1 where different cases will be considered.

Claim 1 *There exist $\lambda_0 \geq 1$, $A_0 > 0$ such that for all $\lambda \geq \lambda_0$*

$$\langle (S_\lambda A_\lambda - A_\lambda S_\lambda) f, f \rangle \geq A_0 \int \varphi_0'' (\lambda^3 f^2 + \lambda^2 f_x^2). \quad (2.12)$$

The proof of this claim is long and it is postponed.

Claim 2 *There exists a constant $C > 0$ which depends on (1.5) and on M_1 , M_2 and M_3 given in (1.2), (1.3), and (1.3) such that*

$$|\langle a(u) (\partial_x f - \lambda \varphi'(x - bt)f), f \rangle| \leq \lambda C. \quad (2.13)$$

Remark 4 *In the above inequality (2.13) the structure of the non-linear term plays a role. This makes a difference with respect to the non-linear Schrödinger equation, see [2].*

Proof Recall that

$$u = e^{-\lambda \varphi(x - bt)} f,$$

so that from (1.3) u is in L^2 . Also observe that

$$e^{\lambda \varphi} \partial_x u = \partial_x f - \lambda \varphi' f.$$

Hence by integration by parts

$$\int a(u) (\partial_x f - \lambda \varphi' f) f dx = \int a(u) e^{\lambda \varphi} \partial_x u e^{\lambda \varphi} u dx = - \int e^{2\lambda \varphi} 2\lambda \varphi' u^2 \alpha(u) dx,$$

with

$$\alpha(s) = \frac{1}{s^2} \int_0^s a(s') s' ds'.$$

The claim easily follows from the boundedness of φ'_0 and (1.2)-(1.5).

Claim 3 *There exists a universal constant C_1 such that*

$$\|a(u) (f_x - \lambda \varphi' f)\|_{L^2}^2 \leq C_1 \frac{M_1^2}{\lambda} \left[(M_2^{1/2} M_3^{1/2} + M_3)^{j/2} + 1 \right] \int \varphi''_0 (\lambda^3 f^2 + \lambda f_x^2) dx. \quad (2.14)$$

Proof We have

$$\begin{aligned} e^{\frac{1}{2}|x-bt|} u^2 &\leq \int_{-\infty}^{\infty} \left| \frac{d}{dy} \left(e^{\frac{1}{2}|y-bt|} u^2(y) \right) \right| dy \\ &\leq 2 \|u_x\|_{L^2} \left\| e^{\frac{1}{2}|y-bt|} u \right\|_{L^2} + \frac{1}{2} \left\| e^{\frac{1}{2}|y-bt|} u \right\|_{L^2}^2 \\ &\leq 2 (M_2 M_3)^{1/2} + \frac{1}{2} M_3. \end{aligned} \quad (2.15)$$

Also there is a universal constant C_0 such that

$$e^{-\frac{1}{2}r} \leq C_0 \varphi''_0(r). \quad (2.16)$$

Then we get from (2.15)

$$|a(u)|^2 \leq C_0 \varphi''_0 (2 (M_2 M_3)^{1/2} + \frac{1}{2} M_3). \quad (2.17)$$

As we already said, either $\varphi = \frac{1}{4} \varphi_0$, or $\varphi = \varphi_0$. Hence

$$|f_x - \lambda \varphi' f|^2 \leq 2 |f_x|^2 + 2 \lambda^2 \sup \varphi'_0 |f|^2 \leq C_0 (|f_x|^2 + \lambda^2 |f|^2), \quad (2.18)$$

for another universal constant C_0 . Also recall that $\lambda \geq 1$. The claim follows from (2.17) and (2.18).

Let us finish the proof of the theorem before we prove Claim 1. We follow the arguments in [2], and more concretely those in Proposition 1 and Proposition 2 of that paper.

Take $\lambda \geq \lambda_0$ and such that

$$C_1 \frac{M_1^2}{\lambda} \left[\left(M_2^{1/2} M_3^{1/2} + M_3 \right)^{j/2} + 1 \right] \leq \frac{A_0}{4}.$$

Recall (2.7) so that (2.4) is satisfied with

$$F = a(u)(f_x - \lambda \varphi'(x - bt)f).$$

Then, from (2.12) and (2.14) we get

$$\langle [S_\lambda; A_\lambda] f, f \rangle + 2\langle S_\lambda f, S_\lambda f \rangle - 2\langle F, S_\lambda f \rangle \geq \langle S_\lambda f, S_\lambda f \rangle + \frac{A_0}{2} \int \varphi_0'' (\lambda^3 f^2 + \lambda^2 f_x^2). \quad (2.19)$$

Our first step is to use the above inequality to obtain a uniform estimate for $\langle S_\lambda f, S_\lambda f \rangle$. Take for any $n \in \mathbb{N}$, $\eta_n(t)$ defined on $[n - 1/2, n + 1/2]$ as

$$\eta_n(t) = \frac{1}{2} - |t - n|.$$

Notice that $\eta_n(n \pm 1/2) = 0$, so that the right hand side of (2.6) is lower bounded using (2.19) by

$$\int \eta_n \|S_\lambda f\|_{L^2}^2 dt.$$

On the other hand $|\eta_n'| = 1$ and $\eta_n'' = -\delta(t - n) \leq 0$. Then, using (1.5) and (2.13), the right hand side of (2.5) is bounded above by λC with C depending on M_1, M_2, M_3 , and the bound in (1.5), but not on n .

As a consequence we get

$$\int \eta_n \|S_\lambda f\|_{L^2}^2 dt \leq \lambda C.$$

Hence there exists a sequence of times $T_n \rightarrow \infty$ such

$$\sup_n \|S_\lambda f(\cdot, T_n)\|_{L^2}^2 < +\infty. \quad (2.20)$$

The second and last step is to obtain a global space-time estimate for f and therefore for u . For proving it we use again identities (2.5) and (2.6) with $F = a(u)(f_x - \lambda \varphi'(x - bt)f)$ and with η defined on $[0, T_n]$ regular and positive, and such that $\eta(0) = 0$ and $\eta \equiv 1$ if $t \geq 1$. With this choice of η the right hand side of (2.5) is bounded using (1.5) and (2.13). For the right hand side of (2.6) we use (2.19) and (2.20). Hence we get that

$$\frac{A_0}{2} \int_1^\infty \int \varphi_0'' (\lambda^3 f^2 + \lambda^2 f_x^2) dx dt < +\infty,$$

and as a consequence

$$\int_1^\infty \int e^{2\lambda|x-bt|} |u(x, t)|^2 dx dt < +\infty.$$

This implies that there exists a sequence of times t_n such that

$$\lim_{t_n \rightarrow \infty} \|u(\cdot, t_n)\| = 0.$$

But from the L^2 conservation law $u_0 \equiv 0$. The result follows from the uniqueness property of the initial value problem in H^1 given in [6].

In order to complete the proof of Theorem 1 it remains to prove Claim 1.

Proof of Claim 1 We shall consider five different cases depending on the sign of b and the relation between b and λ^2 . In what follows C_0 will denote a universal constant that can change from line to line. Only a finite number of choices of these constants will be made.

CASE 1. $\frac{b}{3\lambda^2} \geq 1/2$. In this case we shall take $\varphi = \frac{1}{4}\varphi_0$.

Recall (2.2) and (2.3). Then, we have $(\varphi'_0 \leq 2)$

$$\textcircled{5} + \textcircled{6} + \textcircled{12} \geq \frac{1}{16}\lambda b^2 \int \varphi'' f^2. \quad (2.21)$$

On the other hand

$$\textcircled{7} + \textcircled{8} + \textcircled{9} + \textcircled{10} + \textcircled{11} \leq \int \varphi'' f^2 \{C_0\lambda^3 + \lambda C_0 + \lambda b C_0\}. \quad (2.22)$$

Hence taking λ_0^I large enough we get from (2.21)–(2.22) and for $\lambda > \lambda_0^I$ a bound for

$$\lambda b^2 \int \varphi'' f^2 = \frac{\lambda b^2}{4} \int \varphi_0'' f^2 \geq \frac{9}{16}\lambda^5 \int \varphi_0'' f^2. \quad (2.23)$$

We also have

$$\textcircled{4} \leq 6C_0\lambda \int \varphi'' f_x^2.$$

Then using $\textcircled{2}$ and taking λ_0^I large enough we get a bound of

$$\lambda b \int \varphi'' f_x^2. \quad (2.24)$$

Finally $\textcircled{1}$ gives a bound for $(f_{xx})^2$ and $\textcircled{3}$ is positive but worse than $\textcircled{2}$. Putting everything together we get for $\lambda > \lambda_0^I$ that

$$\begin{aligned} \langle [S_\lambda; A_\lambda] f, f \rangle &\geq A_0^I \int \varphi_0'' (\lambda(f_{xx})^2 + \lambda b f_x^2 + \lambda b^2 f^2) dx \\ &\geq A_0^I \int \varphi_0'' (\lambda^3 f^2 + \lambda^2 f_x^2) dx, \end{aligned} \quad (2.25)$$

for some $A_0^I > 0$.

CASE 2. $0 \leq \frac{b}{3\lambda^2} \leq \frac{1}{2}$, and the integrals in $\textcircled{1}, \textcircled{2}, \dots, \textcircled{12}$ will be considered in the region $r \geq 1$. We take $\varphi = \varphi_0$.

Then

$$\textcircled{5} + \textcircled{6} + \textcircled{12} \geq \frac{9}{4}\lambda^5 \int_{r \geq 1} \varphi_0'' f^2; \quad (2.26)$$

while the absolute value of the integral in the region $r \geq 1$ of $\textcircled{7} + \textcircled{8} + \textcircled{9} + \textcircled{10} + \textcircled{11}$ is upper bounded by

$$\int_{r \geq 1} \varphi_0'' f^2 (C_0\lambda^3 + C_0\lambda + C_0\lambda b). \quad (2.27)$$

Hence taking $\lambda_0 \geq \lambda_0^{II}$ for some λ_0^{II} late enough we get from (2.26)–(2.27) a bound for

$$\lambda^5 \int_{r \geq 1} \varphi_0'' f^2. \quad (2.28)$$

Using this time ③ instead of ② (that is also positive) and ① and (2.28) we get for the region $r \geq 1$ a lower bound for the commutator of the type

$$A_0^{II} \int_{r \geq 1} \varphi_0'' (\lambda (f_{xx})^2 + \lambda^3 f_x^2 + \lambda^5 f^2) dx, \quad (2.29)$$

that is better than what we need.

CASE 3. $0 \leq \frac{b}{3\lambda^2} \leq \frac{1}{2}$ and $r \leq 1$. We take $\varphi = \varphi_0$.

We need a cut off function η such that $\eta(r) = r$ if $r \leq 1$, $\eta(r) \equiv 0$ if $r \geq 3/2$ and $\left| \frac{d^k \eta}{dr^k} \right| \leq C_0$ if $k = 0, 1, 2$. Then for $r = |x - bt|$

$$\begin{aligned} \int \eta(r) f^2(x) dx &= \frac{1}{6\lambda^2} \int (3\lambda r^2 - b) (\eta f^2)_{xx} dx \\ &= \frac{1}{6\lambda^2} \int (3\lambda r^2 - b) \{ (2f f_{xx} + 2f_x^2) \eta + \eta'' f^2 + 2\eta' f f' \}. \end{aligned}$$

Notice that $b \geq 0$, so we get, using Cauchy–Schwartz,

$$\begin{aligned} \int \eta \left(f^2 + \frac{b}{3\lambda^2} f_x^2 \right) &\leq \frac{1}{6\lambda^2} \left(\frac{1}{2} \int (3\lambda^2 r^2 - b)^2 f^2 \eta + 2 \int (f_{xx})^2 \eta + 6\lambda \int r^2 f_x^2 \eta \right. \\ &\quad \left. + C_0 \int_{1 \leq r \leq 3/2} f^2 + \lambda f^2 + \frac{1}{\lambda} f_x^2 \right). \end{aligned} \quad (2.30)$$

From the definition of φ_0 we have that the terms ④, ⑧, ⑨, ⑩ and ⑪ are zero and that for $r \leq 3/2$ we have that ⑤ + ⑥ + ⑫ gives

$$\lambda \int_{r \leq 3/2} (3\lambda^2 r^2 - b)^2 \varphi_0'' f^2.$$

Therefore we just have to take care of ⑦. Using (2.30) we get

$$\begin{aligned} 3\lambda^3 \int_{r \leq 1} f^2 &\leq 3\lambda^3 \int f^2 \eta \\ &\leq \frac{1}{4} \lambda \int_{r \leq 3/2} (3\lambda r^2 - b)^2 f^2 + \lambda \int_{r \leq 3/2} (f_{xx})^2 + 3\lambda^2 \int r^2 f_x^2 \eta \\ &\quad + C_0 \left(\lambda^4 \int_{1 \leq r \leq 3/2} f^2 + \lambda^2 \int_{1 \leq r \leq 3/2} f_x^2 \right) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

I_1 is bounded by ⑤ + ⑥ + ⑫ because $\varphi_0'' = 1$ if $r \leq \frac{3}{2}$, I_2 by ①, I_3 by ③, and I_4 by the bound (2.29) obtained in Case 2, and that as we said was better than what we needed in terms of powers of λ . As a consequence for $\lambda \geq \lambda_0^{III}$ and for some λ_0^{III} large enough we get a lower bound for the commutator of the type

$$\int_{r \leq 1} \varphi_0'' (\lambda^3 f^2 + \lambda(f_{xx})^2) dx. \quad (2.31)$$

Finally from

$$\lambda^2 \int f_x^2 \eta = -\lambda^2 \int (f_x f \eta' + f f'' \eta) = \lambda^2 \int \left(\frac{1}{2} f^2 \eta'' - f f'' \eta \right),$$

and (2.31) we prove that there exists A_0^{III} and a lower bound of the type

$$A_0^{III} \int_{r \leq 1} \varphi_0'' (\lambda^3 f^2 + \lambda^2 f_x^2 + \lambda(f_{xx})^2). \quad (2.32)$$

CASE 4. $b \leq 0$ and $r \geq 1$. We take $\varphi = \varphi_0$.

If $b \leq 0$, ② comes with the wrong sign while ⑫ appears with the good one. Therefore we need another identity. We have

$$\begin{aligned} \textcircled{1} + \textcircled{2} + \textcircled{6} &= 9\lambda \int \varphi_0'' (f_{xx})^2 - 6\lambda b \left(\int \varphi_0'' f f_{xx} + \varphi_0''' f_x f \right) + \lambda b^2 \int \varphi_0'' f^2 \\ &= 9\lambda \int \varphi_0'' \left(f_{xx} - \frac{b}{3} f \right)^2 + 3\lambda b \int \varphi_0^{IV} f^2. \end{aligned} \quad (2.33)$$

Notice that for $r \geq 1$ the absolute value of the last term of the above identity

$$|3\lambda b \int \varphi_0^{IV} f^2|,$$

together with the absolute values of ⑦, ⑧, ⑨, ⑩, and ⑪ are upper bounded by

$$C_0 \int_{r \geq 1} \varphi_0'' f^2 (\lambda|b| + \lambda^3 + \lambda).$$

But from ⑤ and ⑫ we get a lower bound of the type

$$(\lambda^5 + |b|\lambda^3) \int_{1 \leq r} \varphi_0'' f^2,$$

that can be taken in the region $1 \leq r \leq \frac{3}{2}$ as

$$(9\lambda^5 + 6|b|\lambda^3) \int_{1 \leq r \leq \frac{3}{2}} \varphi_0'' f^2.$$

Also ③ gives the bound needed for ④. Altogether we get for $\lambda \geq \lambda^{IV}$ with λ^{IV} large enough, that the commutator is lower bounded by

$$A_0^{IV} \int_{r \geq \frac{3}{2}} \varphi_0'' ((\lambda^5 + |b|\lambda^3) f^2 + \lambda^3 f_x^2) + \int_{1 \leq r \leq \frac{3}{2}} \varphi_0'' ((8\lambda^5 + 5|b|\lambda^3) f^2 + 17\lambda^3 f_x^2) \quad (2.34)$$

for some A_0^{IV} and that is better than what we need.

CASE 5. $b \leq 0$ and $r \leq 1$. We take $\varphi = \varphi_0$.

As in Case 3 we need the cut off $\eta \equiv 1$ if $r \leq 1$ and $\eta \equiv 0$ if $r \geq 3/2$. We have $r = |x - bt|$

$$\begin{aligned} \int f_x^2 \eta &= - \int r \partial_x (f_x^2) \eta - \int r f_x^2 \eta' \\ &= - \int 2r f_x f_{xx} \eta - \int r f_x^2 \eta' \\ &= -2 \int \eta r \left(f_{xx} - \frac{b}{3} f \right) f_x - 2 \frac{b}{3} \int \eta r f f_x - \int r \eta' f_x^2. \end{aligned}$$

Hence using Cauchy–Schwarz and some integration by parts

$$\int f_x^2 \eta - \frac{b}{3} \int f^2 \eta \leq \frac{1}{\lambda} \int \eta \left(f_{xx} - \frac{b}{3} f \right)^2 + \lambda \int \eta r^2 f_x^2 + \int r |\eta'| \left(\frac{|b|}{3} f^2 + f_x^2 \right). \quad (2.35)$$

Also for $r \leq 1$ the terms ④, ⑧, ⑨, ⑩, ⑪ are zero. Therefore only ⑦ remains as a negative term. We have

$$\begin{aligned} \int f^2 \eta &= \frac{1}{2} \int r^2 (f^2)_{xx} \eta - 2 \int r \eta' f f_x - \frac{1}{2} \int r^2 \eta'' f^2 \\ &= \int r^2 (f_x^2 + f f_{xx}) \eta - 2 \int r \eta' f f_x - \frac{1}{2} \int r^2 \eta'' f^2 \\ &= \int r^2 f_x^2 \eta + \int r^2 f \left(f_{xx} - \frac{b}{3} f \right) \eta + \frac{b}{3} \int r^2 f^2 \eta \\ &\quad - 2 \int r \eta' f f_x - \frac{1}{2} \int r^2 \eta'' f^2. \end{aligned}$$

Then, using Cauchy–Schwarz

$$\begin{aligned} \int f^2 \eta - \frac{b}{3} \int r^2 f^2 \eta &\leq \int r^2 \eta f_x^2 + \frac{\lambda^2}{2} \int r^4 \eta f^2 + \frac{1}{2\lambda^2} \int \eta \left(f_{xx} - \frac{b}{3} f \right)^2 \\ &\quad + C_0 \int_{1 \leq |r| \leq 3/2} (1 + \lambda) f^2 + \frac{1}{\lambda} f_x^2. \end{aligned}$$

Then, for $r \leq 1$ the absolute value of ⑦ is upper bounded by

$$\begin{aligned} 3\lambda^3 \int_{r \leq 1} f^2 \eta - \lambda^3 b \int r^2 f^2 &\leq 3\lambda^3 \int r^2 \eta f_x^2 + \frac{3}{2} \lambda^5 \int r^4 \eta f^2 + \frac{3}{2} \lambda \int \eta \left(f_{xx} - \frac{b}{3} f \right)^2 \\ &\quad + C_0 \int_{1 \leq |r| \leq 3/2} \lambda^4 f^2 + \lambda^2 f_x^2 \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Hence, we bound I_1 with ③. Notice that even in the region $1 \leq |r| \leq 3/2$ there is no problem using (2.34). For I_2 we use ⑤ and (2.34) because $\frac{3}{2} r^4 < 8$ if $r \leq 3/2$. For I_3 we use (2.33) ($\varphi_0^{IV} = 0$ if $r \leq 3/2$). Finally for I_4 we use (2.34).

As a conclusion we get a lower bound for the commutator of the type

$$\lambda^3 \int_{r \leq 1} f^2 + \lambda^3 |b| \int r^2 f^2 + \lambda \int \eta \left(f_{xx} - \frac{b}{3} f \right)^2 + \lambda^3 \int \eta r^2 f_x^2.$$

Hence using (2.35) we get a bound of $\lambda^2 \int f_x^2 \eta$. Therefore there exists A_0^V such that the commutator is bounded below by

$$A_0^V \int_{r \leq 1} \varphi_0'' (\lambda^3 f^2 + \lambda^2 f_x^2). \quad (2.36)$$

Taking $A_0 = \max \{A_0^I, A_0^{II}, A_0^{III}, A_0^{IV}, A_0^V\}$ we conclude the proof of Claim 1 using (2.25), (2.29), (2.31), (2.34) and (2.36). Notice that a bound of $\lambda \int \varphi_0'' (f_{xx})^2$ is also obtained.

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