Problem 1. [§4.49.1(a)(b)(c)(f)] Apply the Cauchy-Goursat theorem to show that
\[ \int_C f(z) \, dz = 0 \]
when the contour C is the unit circle \( |z| = 1 \) in either direction, and when

a. \( f(z) = \frac{z^2}{z - 3} \);

b. \( f(z) = ze^{-z} \);

c. \( f(z) = \frac{1}{z^2 + 2z + 2} \);

f. \( f(z) = \log(z + 2) \).

Solution.

a. Notice that the given function is analytic everywhere except at \( z = 3 \), since the denominator vanishes there. Hence the function is analytic in an open disc containing the closed unit disc (for example the open disc of radius 2 around 0). Thus by Cauchy-Goursat, the given integral is 0.

b. The given function is entire, since it is the product of two entire functions. Thus by Cauchy-Goursat, the given integral is 0.

c. By the quadratic formula, the polynomial \( z^2 + 2z + 2 \) has roots \( 1 \pm i \). Thus the given function is analytic everywhere except at these two points. Since these points are at distance \( \sqrt{2} \) from 0, the function is analytic in the open disc of radius \( \sqrt{2} \) around 0 and thus the given integral is 0.

f. Notice that \( \log w \) has a branch cut along the negative real axis (including 0). Let the complement of this ray be \( D \). Suppose \( w = g(z) = z + 2 \), which is entire. Thus \( \log(z + 2) \) is analytic in the domain \( g^{-1}(D) \), which is the complement of the ray starting at \( z = -2 \) and traveling along the negative real axis. Since the given contour is contained in \( D \), by Cauchy-Goursat, the given integral is 0.

Problem 2. [§4.49.3] If \( C_0 \) denotes a positively oriented circle \( |z - z_0| = R \), then
\[ \int_{C_0} (z - z_0)^{n-1} \, dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi & \text{when } n = 0. \end{cases} \]
according to Exercise 10(b), Sec. 42. Use that result and the corollary in Sec. 49 to show that if \( C \) is the boundary of the rectangle \( 0 \leq x \leq 3, 0 \leq y \leq 2 \), described in the positive sense, then

\[
\int_C (z - 2 - i)^{n-1} \, dz = \begin{cases} 
0 & \text{when } n = \pm 1, \pm 2, \ldots, \\
2\pi i & \text{when } n = 0.
\end{cases}
\]

**Solution.** Let \( f(z) = (z - (2 + i))^{n-1} \). Then notice that \( f(z) \) is analytic everywhere except (if \( n < 1 \)) at \( 2 + i \) (if \( n \geq 1 \), the function \( f(z) \) is entire). Notice that \( C \) is contained in this region. Also notice that the maximum distance from \( 2 + i \), of any point on \( C \) is strictly less than 10 (I chose 10 here as a large enough number - you could choose a smaller number that works, but as you will see, for the given result, it will not matter). Hence \( f(z) \) is analytic in the closed region consisting of \( C \) and \( C_{11} \), where \( C_{11} \) is the positively oriented contour of radius 11 around \( 2 + i \). Thus by the corollary,

\[
\int_C f(z) \, dz = \int_{C_{11}} f(z) \, dz.
\]

Thus by the result of Exercise 10(b), the result follows.