Problem 1. [§1.2.1a] Verify that \((\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i\).

Solution.

\[
(\sqrt{2} - i) - i(1 - \sqrt{2}i) = \sqrt{2} - i - 1 + i\sqrt{2}i
= \sqrt{2} - 2i - \sqrt{2}
= -2i.
\]

Problem 2. [§1.2.2] Show that

a. \(\Re(iz) = -\Im(z)\);

b. \(\Im(iz) = \Re(z)\).

Solution.

a. Let \(z = x + iy\). Then \(\Re(iz) = \Re(ix - y) = -y\). Also, \(-\Im(z) = -y\). Thus \(\Re(iz) = -\Im(z)\).

b. Let \(z = x + iy\). Then \(\Im(iz) = \Im(ix - y) = x\). Also, \(\Re(z) = x\). Thus \(\Im(iz) = \Re(z)\).

Problem 3. [§1.2.4] Verify that each of the two numbers \(z = 1 \pm i\) satisfies the equation \(z^2 - 2z + 2 = 0\).

Solution. Notice that

\[
(1 + i)^2 - 2(1 + i) + 2 = 1 + 2i - 1 - 2 - 2i + 2 = 0.
\]

Also,

\[
(1 - i)^2 - 2(1 - i) + 2 = 1 - 2i - 1 - 2 + 2i + 2 = 0.
\]

Problem 4. [§1.3.1] Reduce each of these quantities to a real number:

a. \(\frac{1 + 2i}{3 - 4i} + \frac{2 - i}{5i}\).
b. 

\[
\frac{5t}{(1 - i)(2 - i)(3 - i)}.
\]

c. 

\[(1 - i)^4.
\]

**Solution.**

a. 

\[
\frac{1 + 2t}{3 - 4t} + \frac{2 - t}{5t} = \frac{(1 + 2t)(3 + 4t) + (2 - 5t)(-5t)}{9 + 16} + \frac{25}{25}
\]

\[
= \frac{3 + 6t + 4t - 8 + -10t - 5}{25}
\]

\[
= \frac{2}{5}.
\]

b. 

\[
\frac{5t}{(1 - i)(2 - i)(3 - i)} = \frac{5(1 + i)(2 + i)(3 + i)}{2 \times 5 \times 10}
\]

\[
= \frac{-50}{100}
\]

\[
= -\frac{1}{2}.
\]

c. 

\[(1 - i)^4 = (1 - 2i - 1)^2
\]

\[= -4.
\]
Problem 5. [§1.4.1a] Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when $z_1 = 2i$ and $z_2 = \frac{2}{3} - i$.

Solution.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{$z_1 = 2i$, $z_2 = \frac{2}{3} - i$}
\end{figure}

Problem 6. [§1.4.3] Use established properties of moduli to show that when $|z_3| \neq |z_4|$, 
\[
\frac{\Re(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}
\]

Solution. Recall that $\Re(z_1 + z_2) \leq |z_1 + z_2|$. Further, by the triangle inequality, $|z_1 + z_2| \leq |z_1| + |z_2|$. As an application of the triangle inequality, we also showed that $|z_3 + z_4| \geq ||z_3| - |z_4||$. Thus 
\[
\frac{\Re(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}
\]

Problem 7. [§1.4.4] Verify that $\sqrt{2}|z| \geq |\Re z| + |3z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

Solution. Let $z = x + iy$. Notice it is enough to show that $2|z|^2 \geq (|\Re z| + |3z|)^2$, or $2|x + iy|^2 \geq |x|^2 + |y|^2 + 2|x||y|$. Equivalently, we could show that $|x|^2 + |y|^2 + 2|x||y| - 2|x + iy|^2 \leq 0$. However, since $|x + iy|^2 = |x|^2 + |y|^2$ (by Pythagoras’ theorem), we have that $|x|^2 + |y|^2 + 2|x||y| - 2|x + iy|^2 = -|x|^2 - |y|^2 + 2|x||y| = -(|x| - |y|)^2 \leq 0$. \[
\]
Problem 8. [§1.4.5] In each case, sketch the set of points determined by the given condition:

a. \( |z - 1 + i| = 1 \).
b. \( |z + i| \leq 3 \).
c. \( |z - 4i| \geq 4 \).

Solution.

a. \( R = \{ z \in \mathbb{C} \mid |z - 1 + i| = 1 \} \).
b. $S = \{ z \in \mathbb{C} \mid |z + i| \leq 3 \}$.

c. $T = \{ z \in \mathbb{C} \mid |z - 4i| \geq 4 \}$.