Problem 1. [§2.20.2] Show that

a. a polynomial

\[ P(z) = a_0 + a_1 z + \ldots + a_n z^n \quad a_n \neq 0 \]

of degree \( n \) \( (n \geq 1) \) is differentiable everywhere, with derivative

\[ P'(z) = a_1 + 2a_2 z + \ldots + na_n z^{n-1}; \]

b. the coefficients in the polynomial \( P(z) \) in part (a) can be written

\[ a_0 = P(0), \quad a_1 = P'(0), \quad a_2 = \frac{P''(0)}{2!}, \ldots, \quad a_n = \frac{P^{(n)}(0)}{n!}. \]

Solution.

a. We take the derivative term-by-term to find

\[ \frac{d}{dz} a_i z^i = a_i i z^{i-1}. \]

Thus

\[ \frac{d}{dz} P(z) = a_1 + 2a_2 z + \ldots + na_n z^{n-1}. \]

b. Notice that \( P(0) = a_0 + a_1 0 + \ldots + a_n 0^n = a_0 = a_0 \). We show that

\[ P^{(k)}(z) = \frac{k!}{0!} a_k + \frac{(k + 1)!}{1!} a_{k+1} z + \ldots + \frac{n!}{(n-k)!} z^{n-k}. \]

First notice that the result is true for \( k = 0 \). Assume the result for \( k \). Then

\[ \frac{d}{dz} P^{(k+1)}(z) = \frac{d}{dz} \left[ \frac{k!}{0!} a_k + \frac{(k + 1)!}{1!} a_{k+1} z + \ldots + \frac{n!}{(n-k)!} z^{n-k} \right] \]

\[ = \frac{(k + 1)!}{0!} a_{k+1} + \frac{(k + 2)!}{1!} a_{k+2} z + \ldots + \frac{n!}{(n-k+1)!} z^{n-k+1}. \]

This proves the claim. Thus

\[ P^{(k)}(0) = k! a_k \]

and hence

\[ a_k = \frac{P^{(k)}(0)}{k!}. \]

Problem 2. [§2.20.8] Show that \( f'(z) \) does not exist at any point when

a. \( f(z) = \Re z; \)
b. \( f(z) = \Re z \).

**Solution.** We will prove the result for \( f(z) = \Re z \). The proof for \( \Im z \) is similar. Let \( w(t) = it \) for \( t \in \mathbb{R} \). Then consider
\[
\lim_{t \to 0} \frac{f(z + w(t)) - f(z)}{w(t)} = \lim_{t \to 0} \frac{\Re(z + it) - \Re z}{it} \\
= \lim_{t \to 0} \frac{\Re z + \Re it - \Re z}{it} \\
= \lim_{t \to 0} \frac{0}{it} \\
= 0.
\]

So if \( f'(z) \) exists, then at each point it must equal 0. Now consider the curve \( w(t) = t \) for \( t \in \mathbb{R} \). Then,
\[
\lim_{t \to 0} \frac{f(z + w(t)) - f(z)}{w(t)} = \lim_{t \to 0} \frac{\Re(z + t) - \Re z}{t} \\
= \lim_{t \to 0} \frac{\Re z + \Re t - \Re z}{t} \\
= \lim_{t \to 0} \frac{t}{t} \\
= 1.
\]

But if \( f'(z) \) exists, then it must also equal 1 at every point. Thus \( f'(z) \) does not exist at any point.

**Problem 3.** [§2.23.2(b)(c)(d)] Show that \( f'(z) \) and \( f''(z) \) exist everywhere for

b. \( f(z) = e^{-z}e^{-iy} \);

c. \( f(z) = z^3 \);

d. \( f(z) = \cos x \cosh y - i \sin x \sinh y \).

**Solution.**

b. We rewrite \( f(x + iy) = e^{-x} \cos(-iy) + ie^{-x} \sin(-iy) \). Notice that all first and second order partials exist and are continuous for the given component functions. We only need to check the Cauchy-Riemann equations now.

\[
\begin{align*}
    u_x &= -e^{-x} \cos(-iy); \\
    v_y &= -e^{-x} \cos(-iy); \\
    u_y &= e^{-x} \sin(-iy); \\
    v_x &= -e^{-x} \sin(-iy).
\end{align*}
\]

Thus the Cauchy-Riemann equations hold for \( f \) and
\[
f'(z) = u_x + iv_x = -[e^{-x} \cos(-iy) + ie^{-x} \sin(-iy)].
\]

Since \( f'(z) = -f(z) = -u - iv \), and since the Cauchy-Riemann equations hold for \( f = u + iv \), by linearity of partial derivatives, they also hold for \( f'(z) = -f(z) \) and hence \( f''(z) = f(z) \).
c. We rewrite \( f(x + iy) = x^3 - 3xy^2 + i[3x^2y - y^3] \). Since the component functions are polynomials the first and second order partials exist and are continuous. Then,

\[
\begin{align*}
    u_x &= 3x^2 - 3y^2; \\
    v_y &= 3x^2 - 3y^2; \\
    u_y &= -6xy; \\
    v_x &= 6xy.
\end{align*}
\]

Thus we find that the Cauchy-Riemann equations hold and

\[
f'(z) = u_x + iv_x = 3x^2 - 3y^2 + i(-6xy) = 3(x^2 - y^2 + i2xy) = 3z^2.
\]

We now let \( f' = u + iv \) and recalculate:

\[
\begin{align*}
    u_x &= 6x; \\
    v_y &= 6x; \\
    u_y &= -6y; \\
    v_x &= 6y.
\end{align*}
\]

Thus the Cauchy-Riemann equations hold and \( f''(z) = u_x + iv_x \) exists and equals \( 6x + 6y = 6z \).

d. We first recall that the derivatives of \( \sinh t \) and \( \cosh t \) are each other, and as real functions, they are at least twice continuously differentiable. Thus the component functions are twice continuously differentiable. Then,

\[
\begin{align*}
    u_x &= -\sin x \cosh y; \\
    v_y &= -\sin x \cosh y; \\
    u_y &= \cos x \sinh y; \\
    v_x &= -\cos x \sinh y.
\end{align*}
\]

Thus the Cauchy-Riemann equations hold and \( f'(z) \) exists and equals \( u_x + iv_x = -\sin x \cosh y - i\cos x \sinh y \).

We let \( f' = u + iv \) and recalculate:

\[
\begin{align*}
    u_x &= -\cos x \cosh y; \\
    v_y &= -\cos x \cosh y; \\
    u_y &= -\sin x \sinh y; \\
    v_x &= \sin x \sinh y.
\end{align*}
\]

Thus the Cauchy-Riemann equations hold and \( f'' = u_x + iv_x = -\cos x \cosh y + i\sin x \sinh y \).
Problem 4. \([\S 2.23.8]\) Let \( f = u + iv \) be a differentiable function at a nonzero point \( z_0 = r_0 \exp(i\theta_0) \). Use the expressions for \( u_x \) and \( v_x \) found in Exercise 7 and the polar form of the Cauchy-Riemann equations to rewrite
\[
 f'(z_0) = u_x + iv_x
\]
as
\[
 f'(z_0) = e^{i\theta}[u_r + iv_r].
\]

Solution. First, notice that \( f'(z_0) = u_x + iv_x \). Thus
\[
 f'(z_0) = u_r \cos \theta - u_\theta \frac{\sin \theta}{r} + i[v_r \cos \theta - v_\theta \frac{\sin \theta}{r}].
\]
By the polar form of the Cauchy-Riemann equations,
\[
 ru_r = v_\theta \text{ and } u_\theta = -rv_r.
\]
Thus we find that
\[
 f'(z_0) &= u_r \cos \theta - u_\theta \frac{\sin \theta}{r} + i[v_r \cos \theta - v_\theta \frac{\sin \theta}{r}]
\]
\[
 &= u_r \cos \theta - (rv_r) \frac{\sin \theta}{r} + i[v_r \cos \theta - (ru_r) \frac{\sin \theta}{r}]
\]
\[
 &= u_r \cos \theta + v_r \sin \theta + rv_r \cos \theta - ru_r \sin \theta
\]
\[
 &= u_r \cos \theta - i v_r \sin \theta + i rv_r \sin \theta - i ru_r \cos \theta
\]
\[
 &= u_r [\cos \theta - i \sin \theta] + i v_r [\cos \theta - i \sin \theta]
\]
\[
 &= u_r [\cos \theta - i \sin \theta] + v_r [\cos - \theta + i \sin \theta]
\]
\[
 &= u_r [e^{-i\theta}] + v_r [e^{-i\theta}]
\]
\[
 = e^{-i\theta}[u_r + iv_r].
\]