SECTION 37

2.  (a) \[\int_1^2 \left( \frac{1}{t} - i \right)^2 dt = \int_1^2 \left( \frac{1}{t^2} - 1 \right) dt - 2i \int_1^2 \frac{dt}{t} = -\frac{1}{2} - 2i \ln 2 = -\frac{1}{2} - i \ln 4;\]

\[
(b) \quad \int_0^{\pi/6} e^{2it} dt = \left[ \frac{e^{2it}}{2i} \right]_0^{\pi/6} = \frac{1}{2i} \left[ \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} - 1 \right] = \frac{\sqrt{3}}{4} + i \frac{i}{4};
\]

(c)  Since \( |e^{-bz}| = e^{-bx} \), we find that

\[
\int_0^\infty e^{-zt} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-zt} dt = \lim_{b \rightarrow \infty} \left[ \frac{e^{-zt}}{-z} \right]_0^b = \frac{1}{z} \lim_{b \rightarrow \infty} (1 - e^{-zb}) = \frac{1}{z} \text{ when } \text{Re} z > 0.
\]

3.  The problem here is to verify that

\[
\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}
\]

To do this, we write

\[I = \int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} e^{i(m-n)\theta} d\theta
\]

and observe that when \( m \neq n \),

\[I = \left[ \frac{e^{i(m-n)\theta}}{i(m-n)} \right]_0^{2\pi} = \frac{1}{i(m-n)} - \frac{1}{i(m-n)} = 0.
\]

When \( m = n \), \( I \) becomes

\[I = \int_0^{2\pi} d\theta = 2\pi;
\]

and the verification is complete.

4.  First of all,

\[\int_0^\pi e^{(1+i)x} dx = \int_0^\pi e^x \cos x dx + i \int_0^\pi e^x \sin x dx.
\]

But also,

\[\int_0^\pi e^{(1+i)x} dx = \left[ \frac{e^{(1+i)x}}{1+i} \right]_0^\pi = \frac{e^\pi - 1}{1+i} = \frac{e^\pi - 1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1+e^\pi}{2} + i \frac{1+e^\pi}{2}.
\]
Equate the real parts and then the imaginary parts of these two expressions, we find that
\[ \int_0^\pi e^x \cos x \, dx = -\frac{1 + e^\pi}{2} \quad \text{and} \quad \int_0^\pi e^x \sin x \, dx = \frac{1 + e^\pi}{2}. \]

5. Consider the function \( w(t) = e^t \) and observe that
\[ \int_0^{2\pi} w(t) \, dt = \int_0^{2\pi} e^t \, dt = \left[ \frac{e^t}{i} \right]_0^{2\pi} = \frac{1}{i} - \frac{1}{i} = 0. \]

Since \( |w(c)(2\pi - 0)| = |e^c|2\pi = 2\pi \) for every real number \( c \), it is clear that there is no number \( c \) in the interval \( 0 < t < 2\pi \) such that
\[ \int_0^{2\pi} w(t) \, dt = w(c)(2\pi - 0). \]

6. (a) Suppose that \( w(t) \) is even. It is straightforward to show that \( u(t) \) and \( v(t) \) must be even. Thus
\[ \int_{-\pi}^{\pi} w(t) \, dt = \int_{-\pi}^{\pi} u(t) \, dt + i \int_{-\pi}^{\pi} v(t) \, dt = 2 \int_0^{\pi} u(t) \, dt + 2i \int_0^{\pi} v(t) \, dt \]
\[ = 2 \left[ \int_0^{\pi} u(t) \, dt + i \int_0^{\pi} v(t) \, dt \right] = 2 \int_0^{\pi} w(t) \, dt. \]

(b) Suppose, on the other hand, that \( w(t) \) is odd. It follows that \( u(t) \) and \( v(t) \) are odd, and so
\[ \int_{-\pi}^{\pi} w(t) \, dt = \int_{-\pi}^{\pi} u(t) \, dt + i \int_{-\pi}^{\pi} v(t) \, dt = 0 + i0 = 0. \]

7. Consider the functions
\[ P_n(x) = \frac{1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos \theta\right)^n \, d\theta \quad (n = 0, 1, 2, \ldots), \]
where \(-1 \leq x \leq 1\). Since
\[ |x + i\sqrt{1-x^2}\cos \theta| = \sqrt{x^2 + (1-x^2)\cos^2 \theta} \leq \sqrt{x^2 + (1-x^2)} = 1, \]
it follows that
\[ |P_n(x)| \leq \frac{1}{\pi} \int_0^\pi |x + i\sqrt{1-x^2}\cos \theta| \, d\theta \leq \frac{1}{\pi} \int_0^\pi d\theta = 1. \]
SECTION 40

1. (a) Let $C$ be the semicircle $z = 2e^{i\theta}$ ($0 \leq \theta \leq \pi$), shown below.

Then

$$
\int_C \frac{z + 2}{z} dz = \int_C \left(1 + \frac{2}{z}\right) dz = \int_0^{\pi} \left(1 + \frac{2}{2e^{i\theta}}\right) 2ie^{i\theta} d\theta = 2i \int_0^{\pi} (e^{i\theta} + 1) d\theta
$$

$$
= 2i \left[ \frac{e^{i\theta}}{i} + \theta \right]_0^{\pi} = 2i(i + \pi + i) = -4 + 2\pi i.
$$

(b) Now let $C$ be the semicircle $z = 2e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$) just below.

This is the same as part (a), except for the limits of integration. Thus

$$
\int_C \frac{z + 2}{z} dz = 2i \left[ \frac{e^{i\theta}}{i} + \theta \right]_\pi^{2\pi} = 2i(-i + 2\pi - i - \pi) = 4 + 2\pi i.
$$

(c) Finally, let $C$ denote the entire circle $z = 2e^{i\theta}$ ($0 \leq \theta \leq 2\pi$). In this case,

$$
\int_C \frac{z + 2}{z} dz = 4\pi i,
$$

the value here being the sum of the values of the integrals in parts (a) and (b).

2. (a) The arc is $C: z = 1 + e^{i\theta}$ ($\pi \leq \theta \leq 2\pi$). Then

$$
\int_C (z - 1) dz = \int_{\pi}^{2\pi} (1 + e^{i\theta} - 1)ie^{i\theta} d\theta = i \int_{\pi}^{2\pi} e^{i3\theta} d\theta = i \left[ \frac{e^{i3\theta}}{3i} \right]_{\pi}^{2\pi}
$$

$$
= \frac{1}{2}(e^{i4\pi} - e^{i2\pi}) = \frac{1}{2}(-1 - 1) = 0.
$$
(b) Here \( C: z = x \) \((0 \leq x \leq 2)\). Then
\[
\int_C (z-1)\,dz = \int_0^2 (x-1)\,dx = \left[ \frac{x^2}{2} - x \right]_0^2 = 0.
\]

3. In this problem, the path \( C \) is the sum of the paths \( C_1, C_2, C_3, \) and \( C_4 \) that are shown below.

![Diagram of paths](image)

The function to be integrated around the closed path \( C \) is \( f(z) = \pi e^{\pi z} \). We observe that \( C = C_1 + C_2 + C_3 + C_4 \) and find the values of the integrals along the individual legs of the square \( C \).

(i) Since \( C_1 \) is \( z = x \) \((0 \leq x \leq 1)\),
\[
\int_{C_1} \pi e^{\pi z} \,dz = \pi \int_0^1 e^{\pi x} \,dx = e^\pi - 1.
\]

(ii) Since \( C_2 \) is \( z = 1 + iy \) \((0 \leq y \leq 1)\),
\[
\int_{C_2} \pi e^{\pi z} \,dz = \pi \int_0^1 e^{\pi(1-iy)} \,i\,dy = e^\pi \pi i \int_0^1 e^{-i\pi y} \,dy = 2e^\pi.
\]

(iii) Since \( C_3 \) is \( z = (1-x) + i \) \((0 \leq x \leq 1)\),
\[
\int_{C_3} \pi e^{\pi z} \,dz = \pi \int_0^1 e^{\pi((1-x)-i)}(-1)\,dx = \pi e^\pi \int_0^1 e^{-\pi x} \,dx = e^\pi - 1.
\]

(iv) Since \( C_4 \) is \( z = i(1-y) \) \((0 \leq y \leq 1)\),
\[
\int_{C_4} \pi e^{\pi z} \,dz = \pi \int_0^1 e^{-\pi(1-y)i}(-i)\,dy = \pi i \int_0^1 e^{i\pi y} \,dy = -2.
\]

Finally, then, since
\[
\int_C \pi e^{\pi z} \,dz = \int_{C_1} \pi e^{\pi z} \,dz + \int_{C_2} \pi e^{\pi z} \,dz + \int_{C_3} \pi e^{\pi z} \,dz + \int_{C_4} \pi e^{\pi z} \,dz,
\]
we find that
\[
\int_C \pi e^{\pi z} \,dz = 4(e^\pi - 1).
\]
4. The path \( C \) is the sum of the paths

\[ C_1: z = x + ix^3 \quad (-1 \leq x \leq 0) \quad \text{and} \quad C_2: z = x + ix^3 \quad (0 \leq x \leq 1). \]

Using

\[ f(z) = 1 \text{ on } C_1 \quad \text{and} \quad f(z) = 4y = 4x^3 \text{ on } C_2, \]

we have

\[
\int_{c_1} f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz = \int_{-1}^{0} (1 + i3x^2)dx + \int_{0}^{1} 4x^3(1 + i3x^2)dx \\
= \int_{-1}^{0} dx + 3i \int_{-1}^{0} x^2dx + 4 \int_{0}^{1} x^3dx + 12i \int_{0}^{1} x^5 dx \\
= [x]_{-1}^{0} + i[x^3]_{-1}^{0} + [x^4]_{0}^{1} + 2i[x^6]_{0}^{1} = 1 + i + 1 + 2i = 2 + 3i.
\]

5. The contour \( C \) has some parametric representation \( z = z(t) \quad (a \leq t \leq b) \), where \( z(a) = z_1 \) and \( z(b) = z_2 \). Then

\[
\int_{c} dz = \int_{a}^{b} z'(t)dt = [z(t)]_{a}^{b} = z(b) - z(a) = z_2 - z_1.
\]

6. To integrate the branch

\[ z^{-1+i} = e^{(-1+i)\log z} \quad (|z| > 0, 0 < \arg z < 2\pi) \]

around the circle \( C: z = e^{i\theta} \quad (0 \leq \theta \leq 2\pi) \), write

\[
\int_{c} z^{-1+i} dz = \int_{c} e^{(-1+i)\log z} dz = \int_{0}^{2\pi} e^{(-1+i)(\ln 1 + i\theta)} ie^{i\theta} d\theta = i \int_{0}^{2\pi} e^{-i\theta} e^{i\theta} d\theta = i \int_{0}^{2\pi} e^{-\theta} d\theta = i(1 - e^{-2\pi}).
\]

7. Let \( C \) be the positively oriented circle \( |z| = 1 \), with parametric representation \( z = e^{i\theta} \quad (0 \leq \theta \leq 2\pi) \), and let \( m \) and \( n \) be integers. Then

\[
\int_{c} z^{m} \bar{z}^{n} dz = \int_{0}^{2\pi} (e^{i\theta})^{m}(e^{-i\theta})^{n} ie^{i\theta} d\theta = i \int_{0}^{2\pi} e^{i(m+1)\theta} e^{-i\theta} d\theta.
\]

But we know from Exercise 3, Sec. 37, that

\[
\int_{0}^{2\pi} e^{i\theta} e^{-i\theta} d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}
\]
Consequently,
\[ \int_C z^n \overline{z}^n \, dz = \begin{cases} 0 & \text{when } m+1 \neq n, \\ 2\pi i & \text{when } m+1 = n. \end{cases} \]

8. Note that $C$ is the right-hand half of the circle $x^2 + y^2 = 4$. So, on $C$, $x = \sqrt{4 - y^2}$. This suggests the parametric representation $C: z = \sqrt{4 - y^2} + iy (-2 \leq y \leq 2)$, to be used here. With that representation, we have

\[
\int_C \overline{z} \, dz = \int_{-2}^{2} \left( \sqrt{4 - y^2} - iy \right) \left( \frac{-y}{\sqrt{4 - y^2} + i} \right) dy
\]

\[
= \int_{-2}^{2} (y + y) dy + i \int_{-2}^{2} \left( \frac{y^2}{\sqrt{4 - y^2} + \sqrt{4 - y^2}} \right) dy
\]

\[
= i \int_{-2}^{2} \frac{y^2 + 4 - y^2}{\sqrt{4 - y^2}^2} dy = 4i \int_{-2}^{2} \frac{dy}{\sqrt{4 - y^2}} = 4i \left[ \sin^{-1} \left( \frac{y}{2} \right) \right]_{-2}^{2}
\]

\[
= 4i \left[ \sin^{-1}(1) - \sin^{-1}(-1) \right] = 4i \left[ \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right] = 4\pi i.
\]

10. Let $C_0$ be the circle $z = z_0 + Rei^\theta (-\pi \leq \theta \leq \pi)$.

(a) $\int_{C_0} \frac{dz}{z - z_0} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} Re^{i\theta} d\theta = i \int_{-\pi}^{\pi} d\theta = 2\pi i.$

(b) When $n = \pm 1, \pm 2, \ldots$,

\[
\int_{C_0} (z - z_0)^{n-1} dz = \int_{-\pi}^{\pi} \left( Re^{i\theta} \right)^{n-1} Re^{i\theta} d\theta = iR^n \int_{-\pi}^{\pi} e^{in\theta} d\theta
\]

\[
= \frac{R^n}{n} (e^{in\pi} - e^{-in\pi}) = i \frac{2R^n}{n} \sin n\pi = 0.
\]

11. In this case, where $a$ is any real number other than zero, the same steps as in Exercise 10(b), with $a$ instead of $n$, yield the result

\[
\int_{C_0} (z - z_0)^{a-1} dz = i \frac{2R^a}{a} \sin(a\pi).
\]