1. Hints to previous problems

I will accept solutions to these extra credit problems next week as well for half-credit.

**Problem 1.** Let $K$ be the Cantor set as defined in the notes. Show that the set $K + K = \{ x + y \mid x, y \in K \} = [0, 2]$.

*Solution.* Notice that for each $x \in K$, $x$ has a ternary representation $(0.x_1, x_2, \ldots)$ where no $x_i$ is 1. If we choose any element of $[0, 2]$, either that element $y \in [0, 1)$ or in $[1, 2]$. If it is in $[0, 1)$, then $y = (0.y_1, y_2, \ldots)$. Try and find a way to add up two cantor set numbers to get $y$ by adding two ternary numbers that contain no 1’s.

**Problem 2.** In the notes about the Cantor set, we showed that the ‘length’ of the Cantor set is $0$. However, it is possible to construct a non-empty closed perfect set that contains no intervals and has positive length. Prove that there exists such a set $K'$ with length $\frac{1}{2}$ [Hint: Construct a Cantor-like set that contains no intervals, but the total length of the removed intervals is $\frac{1}{2}$. Then you know that since the length of Cantor-like set + length of removed stuff = 1, the length of the Cantor-like set must be $\frac{1}{2}$. Knowing the geometric series summation formula will help].

*Solution.* Search for ‘fat cantor set’ on google and look at the first wikipedia page. I think it’s called the Smith-Voltera set or something like that. Remember to show that your set is closed, perfect, contains no intervals and the sum of the lengths of the removed intervals is $\frac{1}{2}$.

3. New problem

So we will show that not every set can be given a ‘length’ in a two part extra credit. Let $P \subset \mathcal{P}(\mathbb{R})$ s.t.

1. $\emptyset \in P$,
2. if $A \in P$ then $\mathbb{R} \setminus A \in P$,
3. if $\{A_i\}_{i \in \mathbb{N}}$ is a countable collection of elements of $P$, then for $A = \bigcup_{n \in \mathbb{N}} A_n$, $A \in P$.

Suppose $\ell: P \rightarrow [0, \infty]$ (notice that $\infty$ is in the range) s.t.

1. $\ell(\emptyset) = 0$,
2. for $\{A_i\}_{i \in \mathbb{N}}$ a countable collection of elements of $P$, s.t. for $i \neq j$, $A_i \cap A_j = \emptyset$, $\ell(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \ell(A_i)$ (makes sense since $\bigcup_{i \in \mathbb{N}} A_i \in P$).

Then we call $\ell$ a ‘length’ or ‘measure’ function. We usually want that for $(a, b)$, the interval (closed, open, half-open - doesn’t matter), $\ell(a, b) = b - a$, i.e. we want the length function to make sense on intervals. So let’s assume that for all $a < b \in \mathbb{R}$, $(a, b) \in P$ and $\ell(a, b) = b - a$. We will show that there exists $X \subset \mathbb{R}$ s.t. $\ell(X) \neq 0$ and $\ell(X) \neq 0$ and thus $X$ is not in the domain of $\ell$, i.e. $X$ is a set for which length doesn’t make sense.

First some preliminaries.
Problem 3. Show that every set \{a\} consisting of one point has length 0 (notice that it is the interval \[a, a\]). Then show that every countable set has length 0 (use countable unions).

Let \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\) and let \(Y = [0, 1) \subset \mathbb{R}\). Define \(T : Y \rightarrow Y\) as \(Y(x) = (x + \alpha) - \lfloor (x + \alpha) \rfloor\) (\(x + \alpha\) minus the floor of \(x + \alpha\)). Notice that \(Y(x) \in [0, 1)\). Pick \(x_0 \in [0, 1)\). Let \(T^2(x_0) = T(T(x_0))\) and \(T^3(x_0) = T(T(T(x_0)))\) and thus \(T^n(x_0) = T(T^{n-1}(x_0))\).

Problem 4. Show that the set \(\{x_0, T(x_0), T^2(x_0), \ldots\}\) has length 0 [Hint: the set is countable].

Problem 5. Show that for \(x_0 \in \mathbb{R}\) and \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\), \(T^n(x_0) \neq x_0\) for any \(n \in \mathbb{N}\) [Hint: suppose \(T^n(x_0) = x_0\). Then \((x_0 + n\alpha) - x_0\) is an integer - show why].

Now consider the set \(R_{x_0} = \{\ldots T^{-2}(x_0), T^{-1}(x_0), x_0, T(x_0), T^2(x_0), \ldots\}\).

Problem 6. Show that \(R_{x_0} = R_{x_1}\) if and only if there exists \(z \in \mathbb{Z}\) s.t. \(T^z(x_0) = x_1\) [Hint: \(z \in \mathbb{Z}\) could be negative].

We have thus shown that for any two points \(x_0, x_1 \in [0, 1)\), either \(R_{x_0} \cap R_{x_1} = \emptyset\) or \(R_{x_0} = R_{x_1}\). Thus we can partition \([0, 1)\) into a bunch of sets \(\{R_{x_0} \mid \beta \in J, R_{x_0} \cap R_{x_0'} \neq \emptyset\}\) if and only if \(R_{x_0} = R_{x_0'}\). Let \(X\) be a set that contains exactly one element from each \(R_{x_0}\) in the partition.

Problem 7. Show that the collection \(X \cup T(X) \cup T^2(X) \ldots = [0, 1)\).

We will complete the proof next week. We will show that \(\ell(T(X)) = \ell(X)\). So if \(\ell(X) = 0\), then \(\ell(T^n(X)) = 0\). Since \([0, 1) = \bigcup_{n \in \mathbb{N}} T^n(X)\), \([0, 1)\) will have length 0 if \(X\) has length 0, which is impossible. But if \(\ell(X) = c > 0\), then \(\sum_{n \in \mathbb{N}} \ell(T^n(X)) = \sum_{n \in \mathbb{N}} c = \infty\), which means that \([0, 1)\) has length \(\infty\), which is impossible and thus \(X\) cannot be assigned a length. There are a few more details we have to fill in before we can understand why \(\ell(T(X)) = \ell(X)\).