Problem 1. \([\S 16.1]\)

Solution.

a. False. Let \(s_n = (-1)^n\). Then \(s_1 = s_3\).

b. True. By definition of \(s_n \to s\).

c. False. Let \(s_n = \frac{1}{n}\) and \(t_n = -\frac{1}{n}\). Then \(s_n \to 0\) and \(t_n \to 0\) but \(s_n \neq t_n\).

d. True. Suppose \(s_n \to s\). Then for \(\epsilon = 1\), there exists \(N\) s.t. for all \(n > N\),
\[
|s_n - s| < \epsilon = 1.
\]
Thus \(s_1 < s_n < s + 1\) for all \(n > N\). Let \(a = \min\{s_1, \ldots, s_N, s - 1\}\) and let \(b = \max\{s_1, \ldots, s_N, s + 1\}\). Notice that \(a - 1 < s_n < b + 1\) for all \(n \in \mathbb{N}\).

Problem 2. \([\S 16.2]\)

Solution.

a. True. Since \(s_n \to 0\), for all \(\epsilon > 0\), there exists \(N\) s.t. \(n > N \Rightarrow |s_n - 0| < \epsilon\) and thus \(s_n < \epsilon\).

b. False. Let \(s_n = -1\).

c. False. Let \(s_n = a_n = 1\), let \(s = 0.5\) and let \(k = 1\). Then \(|s_n - 0.5| = |1 - 0.5| = 0.5 \leq 1|a_n| = 1\). However, \(s_n \not\to 0\).

d. True. If \(s \neq t\), then \(|s - t| > 2\epsilon > 0\). However, there exists \(N\) s.t. for all \(n > N\), \(|s_n - s| < \epsilon\). By the triangle inequality, \(|s_n - t| = |s_n - s + s - t| \geq |s - t| - |s_n - s| \geq |s - t| - \epsilon = 2\epsilon - \epsilon = \epsilon\). Thus \(s_n \not\to t\).

Problem 3. \([\S 16.7]\)

Solution.

d. Notice that \(0 < \frac{\sqrt{n}}{n+1} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \to 0\). Thus by the squeeze theorem, \(\frac{\sqrt{n}}{n+1} \to 0\).

e. Notice that \(0 < \frac{x^2}{n^2} = \frac{n^2}{(n-1)!} \leq \frac{n}{(n-1)!} = \frac{1}{(n-2)!} \leq \frac{1}{n-2} \to 0\). Thus by the squeeze theorem, \(\frac{x^2}{n^2} \to 0\).

f. Choose \(\epsilon > 0\). Let \(|x| < 1\). Notice that \(\log |x| < 0\). Choose \(N > \frac{\log x}{\log |x|}\) by the archimedean property. Then for all \(n > N\), \(n \log |x| < N \log |x| < \log \epsilon \Rightarrow |x|^n < \epsilon\) (where the first inequality holds since \(\log |x| < 0\)). Thus \(|x|^n \to 0\).
Problem 4. [§16.8]

Solution.

a. Notice that \( a_n = 2n \) is unbounded and thus cannot converge.

b. We will first prove that \( x_n \) takes only finitely many values \( \{ s_1, \ldots, s_k \} \) and if \( x_n \) converges, then \( \lim_{n \to \infty} x_n \in \{ s_1, \ldots, s_k \} \). We know that for some \( i \leq k \) (and this \( i \) need not be unique \textit{a priori}) there exist infinitely many \( n \in \mathbb{N} \) s.t. \( x_n = s_i \). Now suppose \( x_n \) converges to \( t \neq s_i \). Since \( t \neq s_i \), \( |s_i - t| > 2 \varepsilon > 0 \). However, there exists \( N \) s.t. for all \( n > N \), \( |x_n - s_i| < \varepsilon \). By the triangle inequality, \( |x_n - t| = |x_n - s_i + s_i - t| \geq |s_i - t| - |x_n - s_i| \geq |s_i - t| - \varepsilon = 2 \varepsilon - \varepsilon = \varepsilon \). Thus, \( x_n \not\to t \) and thus since \( x_n \) converges and cannot converge to any point \( t \neq s_i \), \( x_n \to s_i \).

Since \( b_n = (-1)^n \), if \( b_n \) converged, it would have to converge to either \(-1\) or \(1\) (since \( b_n \) only takes on the values \(-1, 1\)). However, for all \( N \in \mathbb{N}, \) there exists \( n > N \) s.t. \( |b_n - 1| > \frac{1}{2} \) and there exists \( m > N \) s.t. \( |b_m + 1| > \frac{1}{2} \) and thus \( b_n \) does not converge.

c. As above, if \( c_n \) converged it would have to converge to one of \( \{1, \frac{1}{2}, -\frac{1}{2}, -1\} \). However, as above \( \cos(\frac{n\pi}{4}) \) does not converge to any of the points in that set.

d. \( d_n = (-n)^2 = n^2 \), which is not bounded and thus doesn’t converge.

\( \square \)

Problem 5. [§16.12]

Solution.

a. Suppose that \( t_n \) is bounded. Then there exists \( M > 0 \) s.t. \( |t_n| < M \). Then \( |s_nt_n| = |s_n||t_n| < M|s_n| \). Since \( s_n \to 0 \), by the squeeze theorem, \( s_nt_n \to 0 \).

b. Let \( s_n = \frac{1}{n} \) and let \( t_n = n \). Then \( s_nt_n = 1 \to 1 \), but \( s_n \to 0 \).

\( \square \)

Problem 6. [§16.15]

Solution.

a. \( \Rightarrow \) Suppose \( x \) is an accumulation points of \( S \). Then for all \( \varepsilon > 0 \), \( N^{\ast}(x, \varepsilon) \cap S \neq \emptyset \). Let \( \varepsilon = \frac{1}{n} \) and choose \( x_n \in N^{\ast}(x, \frac{1}{n}) \cap S \). Thus \( x_n \) is a sequence contained in \( S \setminus \{x\} \) s.t. \( 0 < |x_n - x| < \frac{1}{n} \to 0 \). Thus there exists a sequence \( \{x_n \mid n \in \mathbb{N}\} \subset S \setminus \{x\} \) s.t. \( x_n \to x \).

\( \Leftarrow \) Suppose there exists \( \{s_n \mid n \in \mathbb{N}\} \subset S \setminus \{x\} \) s.t. \( s_n \to x \). Then for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) s.t. \( n > N \Rightarrow |s_n - x| < \varepsilon \) and thus \( s_n \in N(x, \varepsilon) \). Since for \( n > N, s_n \in S \setminus \{x\} \) and \( s_n \in N(x, \varepsilon) \), we have that \( s_n \in N^{\ast}(x, \varepsilon) \cap S \) and thus \( N^{\ast}(x, \varepsilon) \cap S \neq \emptyset \) and thus \( x \) is an accumulation points of \( S \).

b. \( \Rightarrow \) Suppose \( S \) is closed. Let \( \{s_n \mid n \in \mathbb{N}\} \subset S \) s.t. \( s_n \to s \). Suppose that for some \( n \in \mathbb{N}, s_n = x \). Then since \( s_n \in S, s \in S \). Else, for all \( n \in \mathbb{N}, s_n \neq s \). Thus \( \{s_n \mid n \in \mathbb{N}\} \subset S \setminus \{x\} \). By part a, we have that \( s \) is a limit point of \( S \). Since \( S \) is closed, \( s \in S \).
‘⇐’ Suppose that whenever \( \{ s_n \mid n \in \mathbb{N} \} \subset S \) is a convergent sequence, then \( \lim_{n \to \infty} s_n = s \in S \). Let \( s \in S' \).

Then by part a, there exists a sequence \( s_n \) s.t. \( s_n \in S \setminus \{ x \} \) and \( s_n \to s \). Thus \( s_n \in S \). By assumption, \( s = \lim_{n \to \infty} s_n \in S \) and thus \( S' \subset S \) and \( S \) is closed.

\( \square \)