MATH 122A HW 1 SOLUTIONS

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Problem 1. [§1.2.1a] *Verify that* $(\sqrt{2}-\imath)-\imath(1-\sqrt{2}\imath)=-2\imath$.

Solution.

$$\begin{aligned}(\sqrt{2}-\imath)-\imath(1-\sqrt{2}\imath) &= \sqrt{2}-\imath-\imath+\imath^2\sqrt{2}\\ &= \sqrt{2}-2\imath-\sqrt{2}\\ &= -2\imath.\end{aligned}$$

Problem 2. [§1.2.2] *Show that*

a. $\Re(\imath z)=-\Im(z)$;

b. $\Im(\imath z)=\Re(z)$.

Solution.

a. Let $z=x+\imath y$. Then $\Re(\imath z)=\Re(\imath x-y)=-y$. Also, $-\Im(z)=-y$. Thus $\Re(\imath z)=-\Im(z)$.

b. Let $z=x+\imath y$. Then $\Im(\imath z)=\Im(\imath x-y)=x$. Also, $\Re(z)=x$. Thus $\Im(\imath z)=\Re(z)$.

Problem 3. [§1.2.4] *Verify that each of the two numbers* $z=1\pm\imath$ *satisfies the equation* $z^2-2z+2=0$.

Solution. Notice that

$$\begin{aligned}(1+\imath)^2-2(1+\imath)+2 &= 1+2\imath-1-2-2\imath+2\\ &= 0.\end{aligned}$$

Also,

$$\begin{aligned}(1-\imath)^2-2(1-\imath)+2 &= 1-2\imath-1-2+2\imath+2\\ &= 0.\end{aligned}$$

Problem 4. [§1.3.1] *Reduce each of these quantities to a real number:*

a.

$$\frac{1+2\imath}{3-4\imath}+\frac{2-\imath}{5\imath};$$

b.

$$\frac{5\imath}{(1-\imath)(2-\imath)(3-\imath)};$$

c.

$$(1-\imath)^4.$$

Solution.

a.

$$\begin{aligned}\frac{1+2\imath}{3-4\imath}+\frac{2-\imath}{5\imath} &= \frac{(1+2\imath)(3+4\imath)}{9+16}+\frac{(2-5\imath)(-5\imath)}{25}\\ &= \frac{3+6\imath+4\imath-8}{25}+\frac{-10\imath-5}{25}\\ &= \frac{2}{5}.\end{aligned}$$

b.

$$\begin{aligned}\frac{5\imath}{(1-\imath)(2-\imath)(3-\imath)} &= \frac{5\imath(1+\imath)(2+\imath)(3+\imath)}{2\times 5\times 10}\\ &= \frac{-50}{100}\\ &= -\frac{1}{2}.\end{aligned}$$

c.

$$\begin{aligned}(1-\imath)^4 &= (1-2\imath-1)^2\\ &= -4.\end{aligned}$$

$\square$

Problem 5. [§1.4.1a] *Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when $z_1 = 2\imath$ and $z_2 = \frac{2}{3} - \imath$.*

Solution. □

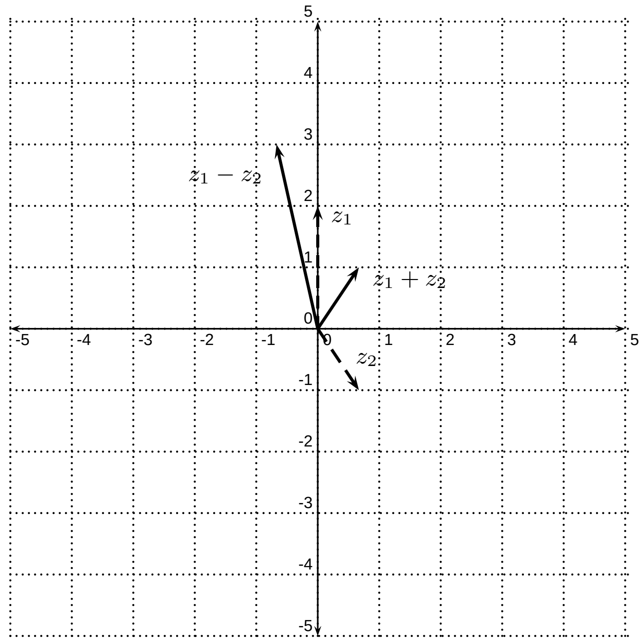


FIGURE 1. $z_1 = 2\imath$, $z_2 = \frac{2}{3} - \imath$

Problem 6. [§1.4.3] *Use established properties of moduli to show that when $|z_3| \neq |z_4|$,*

$$\frac{\Re(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

Solution. Recall that $\Re(z_1 + z_2) \leq |z_1 + z_2|$. Further, by the triangle inequality, $|z_1 + z_2| \leq |z_1| + |z_2|$. As an application of the triangle inequality, we also showed that $|z_3 + z_4| \geq ||z_3| - |z_4||$. Thus

$$\frac{\Re(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

□

Problem 7. [§1.4.4] *Verify that $\sqrt{2}|z| \geq |\Re z| + |\Im z|$.*

Suggestion: *Reduce this inequality to $(|x| - |y|)^2 \geq 0$.*

Solution. Let $z = x + \imath y$. Notice it is enough to show that $2|z|^2 \geq (|\Re z| + |\Im z|)^2$, or $2|x + \imath y|^2 \geq |x|^2 + |y|^2 + 2|x||y|$. Equivalently, we could show that $|x|^2 + |y|^2 + 2|x||y| - 2|x + \imath y|^2 \leq 0$. However, since $|x + \imath y|^2 = |x|^2 + |y|^2$ (by Pythagoras' theorem), we have that $|x|^2 + |y|^2 + 2|x||y| - 2|x + \imath y|^2 = -|x|^2 - |y|^2 + 2|x||y| = -(|x| - |y|)^2 \leq 0$. □

Problem 8. [§1.4.5] *In each case, sketch the set of points determined by the given condition:*

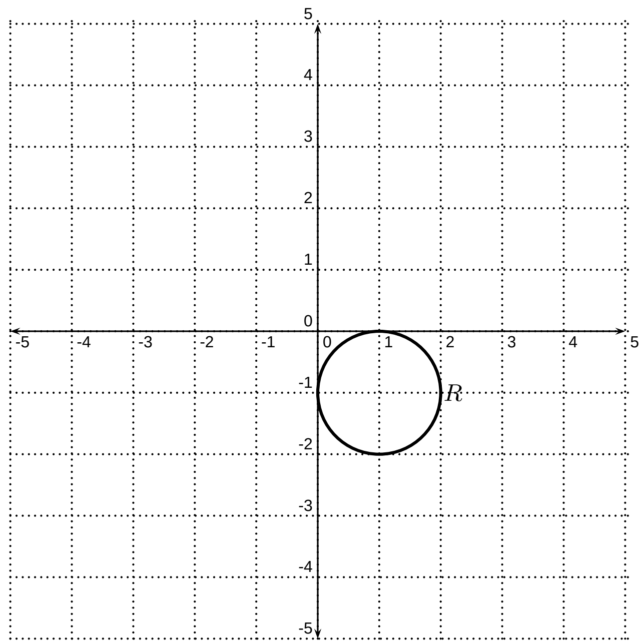
a. $|z - 1 + \imath| = 1.$

b. $|z + \imath| \leq 3.$

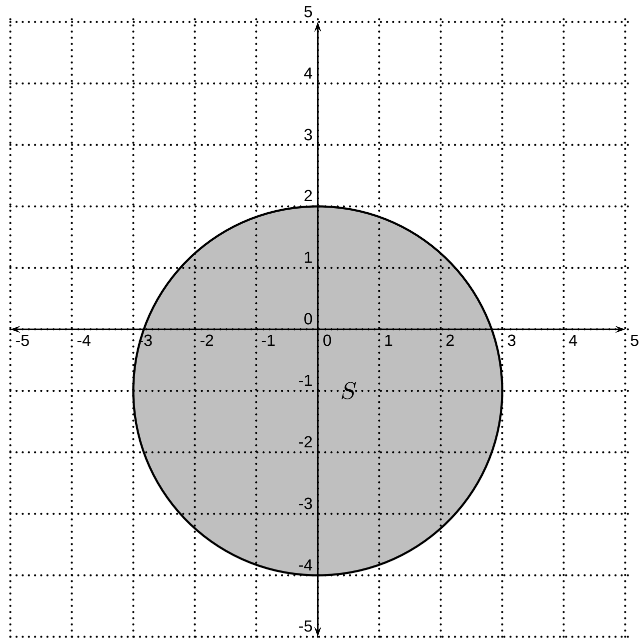
c. $|z - 4\imath| \geq 4.$

Solution.

a. $R = \{z \in \mathbb{C} \mid |z - 1 + \imath| = 1\}.$



b. $S = \{z \in \mathbb{C} \mid |z + \imath| \leq 3\}$.



c. $T = \{z \in \mathbb{C} \mid |z - 4\imath| \geq 4\}$.