A (Hopefully Gentle) Introduction to Arithmetic Kleinian Groups

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Dramatic Reenactment

Blah, Blah, Blah, Arithmetic 3-Manifold, Blah, Blah, Blah
Dramatic Reenactment

Hey Neil, what is an arithmetic 3-manifold?
Dramatic Reenactment

Blah Blah, its complicated, figure eight knot, blah blah
Kleinian Groups

- The group of orientation preserving isometries of $\mathbb{H}^3$ can be identified with $\text{PSL}_2(\mathbb{C})$.
- A discrete subgroup, $\Gamma$, of $\text{PSL}_2(\mathbb{C})$ is called a *Kleinian group*.
- A *hyperbolic 3-orbifold* is the quotient of $\mathbb{H}^3$ by a Kleinian group, denoted $O \cong \mathbb{H}^3 / \Gamma$.
- If $\mathbb{H}^3 / \Gamma$ is compact, we say that $\Gamma$ is *cocompact*.
- If $\mathbb{H}^3 / \Gamma$ has finite volume we say that $\Gamma$ has *finite covolume*. 
Commensurability

- Two Kleinian groups $\Gamma$ and $\Gamma'$ are *commensurable* if $\Gamma \cap \Gamma'$ has finite index in both $\Gamma$ and $\Gamma'$.

- Two Kleinian groups, $\Gamma$ and $\Gamma'$, are *commensurable in the wide sense* if there exists $g \in \text{PSL}_2(\mathbb{C})$ such that $g^{-1}\Gamma g$ and $\Gamma'$ are commensurable.

- If $O \cong \mathbb{H}^3/\Gamma$ and $O' \cong \mathbb{H}^3/\Gamma'$, then this definition is equivalent to $O$ and $O'$ having a common finite sheeted cover.
Some Triviality Conditions

Let $\Gamma$ be a subgroup of $\text{PSL}_2(\mathbb{C})$

- $\Gamma$ is *reducible* if all the elements of $\Gamma$ have a common fixed point, otherwise it is *irreducible*.
- The group is *elementary* if it has a finite orbit in its action on $\mathbb{H}^3 \cup \partial \mathbb{H}^3$, otherwise it is *non-elementary*.
- Notice that reducible groups are elementary, but that elementary groups need not be reducible.

**Remark**

We will often want to rule out groups in these two categories. Almost all the groups we will consider will be irreducible and non-elementary. For example, finite volume Kleinian groups are irreducible and non-elementary.
Some Field Theory

Let $k \setminus \mathbb{Q}$ be a finite extension of fields (i.e. $k$ is a field that is a finite dimensional vector space over $\mathbb{Q}$). If $[k : \mathbb{Q}] = n$ then we have the following facts.

- There exists $t \in k$ such the minimal polynomial of $t$ over $\mathbb{Q}$ has degree $n$ and $k \cong \mathbb{Q}(t)$.
- If $t = t_1, t_2, \ldots, t_n$ are the roots of the minimal polynomial of $t$ then the assignments $t \mapsto t_i$ induces an embedding, denoted $\sigma_i$, of $k$ into $\mathbb{C}$.
- Conversely, any embedding of $k$ into $\mathbb{C}$ must map $t$ to one of the $t_i$ and thus there are exactly $n$ embeddings of $k$ into $\mathbb{C}$.
- If $\sigma_i(k) \subset \mathbb{R}$ then the embedding is called a real place. If $\sigma_i(k) \not\subset \mathbb{R}$ then there exists $j \neq i$ such that $\sigma_j = \overline{\sigma_i}$, and we refer to the pair $(\sigma_i, \sigma_j)$ as a complex place.
- Therefore we arrive at the formula $n = r_1 + 2r_2$ where $r_1$ is the number of real places and $r_2$ is the number of conjugate places of complex embeddings.
Norm and Trace

- Define the **norm of** $\alpha$ to be $N_{k \setminus \mathbb{Q}}(\alpha) := \prod_{i=1}^{n} \sigma_i(\alpha)$. This is a homomorphism from $k$ to $\mathbb{Q}$.
- Define the **trace of** $\alpha$ to be $Tr_{k \setminus \mathbb{Q}}(\alpha) := \sum_{i=1}^{n} \sigma_i(\alpha)$. This is also a homomorphism from $k$ to $\mathbb{Q}$.
- If $k$ is a quadratic imaginary extension of $\mathbb{Q}$ then $N_{k \setminus \mathbb{Q}}(\alpha) = |\alpha|^2$ and $Tr_{k \setminus \mathbb{Q}}(\alpha) = 2\text{Re}(\alpha)$.

**Remark**

In general, the norm and complex modulus of field element are unrelated.
Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$, then an element $\alpha \in \overline{\mathbb{Q}}$ is an \textit{algebraic integer} if its minimal polynomial has entries in $\mathbb{Z}$.

Given a finite extension, $k$, of $\mathbb{Q}$ let the set $\mathcal{O}_k$ be a set of algebraic integers contained in $k$.

Since $N_{k/\mathbb{Q}}(\alpha)$ and $Tr_{k/\mathbb{Q}}(\alpha)$ can be thought of as entries of the minimal polynomial of $\alpha$, we see that if $\alpha \in \mathcal{O}_k$ then $N_{k/\mathbb{Q}}(\alpha)$ and $Tr_{k/\mathbb{Q}}(\alpha)$ are in $\mathbb{Z}$. In a degree 2 extension, the converse is also true.
The Modular Group

- $\text{PSL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C}) \mid a, b, c, d \in \mathbb{Z} \right\}$ is arithmetic.

- If $\Gamma = \text{PSL}_2(\mathbb{Z})$, then $O = \mathbb{H}^3/\Gamma$ is the 2-orbifold with base space the disk and two cone points of order 2 and 3 respectively.

- As we will see, arithmeticity is a property of a commensurability class, and so any finite sheeted cover of $O$ is arithmetic.

- For example, if $\Gamma' := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \ mod \ 2, b \equiv c \equiv 0 \ mod \ 2 \right\}$. $\mathbb{H}^3/\Gamma'$ is the thrice punctured sphere.
The Bianchi Groups

- Let $k = \mathbb{Q}(\sqrt{-d})$, where $d > 0$, then $\text{PSL}_2(\mathcal{O}_k)$ is a Bianchi group. These groups are the 3-dimensional analogue of the modular group.
- The Bianchi groups provide infinitely many commensurability classes of arithmetic, Kleinian groups.
- All 3-dimensional cusped, arithmetic, Kleinian groups are commensurable with a Bianchi group.
A Non-Example

- Let $k$ be an extension of $\mathbb{Q}$, such that $[k : \mathbb{Q}] > 2$, should we consider $\text{PSL}_2(\mathcal{O}_k)$ to be an arithmetic group?

- No, there exists an element $\gamma \in \mathcal{O}_k$ such that $|\gamma| < 1$.

- Therefore, the sequence $\begin{pmatrix} 1 & \gamma^n \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and so $\text{PSL}_2(\mathcal{O}_k)$ is not even a discrete subgroup of $\text{PSL}_2(\mathbb{C})$.

- So we see that our examples already have two problems.

  1. We have no closed examples.
  2. Our coefficients are restricted to lie in an extension of $\mathbb{Q}$ of degree at most 2.
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The Invariant Trace Field

• Let $\Gamma$ be a finite covolume Kleinian group, then the *trace field of* $\Gamma$, denoted $\mathbb{Q}(\text{tr}\Gamma)$, is the field obtained by adjoining the traces of all elements of $\Gamma$ to $\mathbb{Q}$.

• Let $\Gamma^{(2)} = \langle \gamma^2 : \gamma \in \Gamma \rangle$, then $\Gamma^{(2)}$ is a finite index, normal subgroup of $\Gamma$, whose quotient is an Abelian 2-group.

• The trace field of $\Gamma^{(2)}$ is called the *invariant trace field*, and is denoted $k\Gamma$.

**Theorem**

$\mathbb{Q}(\text{tr}\Gamma)$ is an finite extension of $\mathbb{Q}$ and is a topological invariant of $\mathbb{H}^3/\Gamma$.

**Theorem**

$k\Gamma$ is an invariant of the commensurability class of $\Gamma$. 

“Small” Algebraic Integers

A useful fact about algebraic integers is that there are only finitely many “small” algebraic integers. More precisely we have.

**Lemma**

*There are only finitely many algebraic integers of bounded degree all of whose Galois conjugates are bounded.*

**Proof.**

- Let $\alpha$ be such an algebraic integer. The coefficients of the minimal polynomial of $\alpha$ are symmetric functions in $\alpha$ and its conjugates.
- Since all conjugates of $\alpha$ are bounded the entries of the minimal polynomial of $\alpha$ are bounded.
- Since there are only finitely many polynomials of bounded degree with bounded integer coefficients there are only finitely many algebraic integers with the above property.
The Motivating Theorem

Given an arbitrary group $\Gamma \subset \text{PSL}_2(\mathbb{C})$, it is typically difficult to decide whether or not it is discrete. The following theorem shows how we can use arithmetic data to show discreteness.

**Theorem (The Motivating Theorem)**

Let $\Gamma$ be a finitely generated subgroup of $\text{PSL}_2(\mathbb{C})$ such that

1. $\Gamma^{(2)}$ is irreducible.
2. $\text{tr}(\gamma)$ is an algebraic integer for every $\gamma \in \Gamma$.
3. For each embedding, $\sigma : k\Gamma \to \mathbb{C}$, other than $\text{Id}$ and complex conjugation, the set $\{\sigma(\text{tr}(\gamma)) : \gamma \in \Gamma^{(2)}\}$ is bounded.

Then $\Gamma$ is discrete.
A First Definition of Arithmeticity

The previous theorem provides us with our first definition of arithmeticity.

• A group $\Gamma \subset \text{PSL}_2(\mathbb{C})$ is arithmetic if it satisfies the conditions of the Motivating Theorem and has finite covolume.
• Notice that the Bianchi groups are arithmetic under this definition.
• Since $k\Gamma$ is a commensurability invariant we see that if $\Gamma$ is arithmetic and $\Gamma'$ is commensurable with $\Gamma$, then $\Gamma'$ is arithmetic.
• Unfortunately, these conditions are not packaged in the best way. In particular the third one can be difficult to verify.
• Fortunately, there are better ways to view condition 3.
Quatennion Algebras and Hilbert Symbols

- A **quaternion algebra** $A$ over a field $F$ of characteristic $\neq 2$ is a four-dimensional algebra over the field $F$, with basis $\{1, i, j, k\}$ and elements $a, b \in F^*$ such that
  
  $$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$ 

- A quaternion algebra can be described by a (non-unique) **Hilbert symbol**, $\left(\frac{a, b}{F}\right)$, where the entries are squares of a pair of the basis elements other than 1.

- Other possible Hilbert symbols include $\left(\frac{b, a}{F}\right)$ and $\left(\frac{a, -ab}{F}\right)$.

- Another possible Hilbert symbol for $\left(\frac{a, b}{F}\right)$ is $\left(\frac{ax^2, by^2}{F}\right)$, where $x, y \in F^*$, however this involves a basis change to $i' = xi, j' = yj$ and $k' = i'j'$. 
Some Examples

Some familiar examples of quaternion algebras include

- The Hamilton quaternions, denoted \( \mathcal{H} = \left( \frac{-1, -1}{\mathbb{R}} \right) \).

- For any field \( F \), \( M_2(F) = \left( \frac{1,1}{F} \right) \), where \( i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

- Since every element of \( \mathbb{C} \) is a square we see that every quaternion algebra over \( \mathbb{C} \) is \( \left( \frac{1,1}{\mathbb{C}} \right) = M_2(\mathbb{C}) \).

- Similarly, since every positive real is a square, every quaternion algebra over \( \mathbb{R} \) is equal to one of \( \left( \frac{1,1}{\mathbb{R}} \right) \), \( \left( \frac{1,-1}{\mathbb{R}} \right) \), and \( \left( \frac{-1,-1}{\mathbb{R}} \right) \). The first two are \( M_2(\mathbb{R}) \) and the third is \( \mathcal{H} \).
Yet Another Norm and Trace

Quaternion algebras are equipped with a natural notion of norm and trace, similar to the complex numbers.

- Let $A_0$ be the subspace of $A$ spanned by $\{i, j, k\}$. The elements of $A_0$ are called pure quaternions.
- It turns out that this subspace is independent of the choice of basis.
- Every $x \in A$ has a decomposition $x = a + \alpha$ where $a \in F$ and $\alpha \in A_0$.
- Define the conjugate of $x$, denoted $\overline{x}$, by $\overline{x} = x - \alpha$.
- Define the norm and trace of $x$ by $n(x) = x\overline{x}$ and $tr(x) = x + \overline{x}$, respectively.
Ramified Vs. Unramified Places

- Let $k$ be a finite extension of $\mathbb{Q}$, let $A = \left( \frac{a,b}{k} \right)$ be a quaternion algebra, and let $\sigma$ be an embedding of $k$ into $\mathbb{C}$.
- If $\sigma$ is real place then
  
  \[ A \otimes_{\sigma(k)} \mathbb{R} \cong \left( \frac{a,b}{k} \right) \otimes_{\sigma(k)} \mathbb{R} \cong \left( \frac{\sigma(a), \sigma(b)}{\mathbb{R}} \right). \]

- Let $\iota : A \rightarrow A \otimes_{\sigma(k)} \mathbb{R}$ (or $\mathbb{C}$) be the canonical injection, then for $x \in A$, $\sigma(tr(x)) = tr(\iota(x))$ and $\sigma(n(x)) = n(\iota(x))$.
- If $A \otimes_{\sigma(k)} \mathbb{R}$ is isomorphic to $\mathcal{H}$ then $A$ is ramified at $\sigma$, and if $A$ is isomorphic to $M_2(\mathbb{R})$ then $A$ is unramified at $\sigma$.
- If $\sigma$ is a complex place then $A \otimes_{\sigma(k)} \mathbb{C} \cong M_2(\mathbb{C})$, and so $A$ is unramified at every complex place.
Norm and Trace in the Ramified Case

In \( \mathcal{H} \) there is a strong connection between norm and trace that is absent in \( M_2(\mathbb{R}) \).

- Let \( x \in \mathcal{H} \), then \( x = a + bi + cj + dk \).
- Therefore,
  \[
  n(x) = a^2 + b^2 + c^2 + d^2, \quad \text{and} \quad tr(x) = 2a.
  \]
- Thus if \( n(x) = 1 \), then \( tr(x) \in [-2, 2] \).
- Conversely, in \( M_2(\mathbb{R}) \) there are elements with norm 1 whose trace is arbitrarily large.
The Invariant Quaternion Algebra

Let $\Gamma$ be a finitely generated, non-elementary subgroup of $\text{PSL}_2(\mathbb{C})$, and define the *invariant quaternion algebra*, denoted $A\Gamma$ by,

$$A\Gamma = \left\{ \sum a_i \gamma_i \mid a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)} \right\}.$$  

**Theorem**

$A\Gamma$ is a quaternion algebra over $k\Gamma$.

**Theorem**

$A\Gamma$ is an invariant of the commensurability class of $\Gamma$. 
Some Properties of $A\Gamma$

- $\Gamma^{(2)}$ is naturally embedded inside of $A\Gamma$.
- In $A\Gamma$, the trace of $\gamma$ as a matrix agrees with its trace as an element of the quaternion algebra.
- Also, since the determinant of $\gamma \in \Gamma$ is 1 we see that all elements of $\Gamma$ have norm 1.
- We can now rephrase the Motivating Theorem in terms of quaternion algebras.
Restatement of the Motivating Theorem

Theorem

If $\Gamma$ is a finitely generated subgroup of $\text{PSL}_2(\mathbb{C})$ satisfying conditions 1 and 2 of the Motivating Theorem, then condition 3 is equivalent to the following

$3'$ All embeddings, $\sigma$, from $k\Gamma$ to $\mathbb{C}$ other than $\text{Id}$ and complex conjugation are real and $A\Gamma$ is ramified at all real places.

Proof.

• If $3'$ holds and $\sigma$ is a real embedding other than $\text{Id}$ or conjugation, then we can find $\nu : A\Gamma \to \mathcal{H}$ such that for $\gamma \in \Gamma^{(2)}$ we have $\sigma(tr(\gamma)) = tr(\nu(\gamma))$.

• Since $n(\nu(\gamma)) = 1$, we have previously seen that $tr(\nu(\gamma)) \in [-2, 2]$, and thus $\sigma(tr(\gamma)) \in [-2, 2]$.

• Conversely, if 3 is true then it can be shown that $A\Gamma \otimes_{\sigma(k)} \mathbb{R}$ has a Hilbert symbol, \( \left( \frac{a,b}{\mathbb{R}} \right) \), where $a, b < 0$, and so $A\Gamma$ is ramified at all real places.
Orders

Let $\Gamma$ be arithmetic and let

$$A\Gamma \supset \mathcal{O} = \left\{ \sum a_i \gamma_i \mid a_i \in \mathcal{O}_{k\Gamma}, \gamma_i \in \Gamma^{(2)} \right\} \text{ and } \mathcal{O}^{-1} = \left\{ x \in \mathcal{O}^{-} \mid n(x) = 1 \right\}$$

- Since $\Gamma \subset \text{PSL}_2(\mathbb{C})$ we know that $A\Gamma \subset M_2(\mathbb{C})$.
- Let $\sigma$ be the unique complex place of $k\Gamma$, then we have a map $\iota : A\Gamma \to A \otimes_{\sigma(k\Gamma)} \mathbb{C} \cong M_2(\mathbb{C})$.
- By the Skolem-Noether Theorem, there exists $g \in GL_2(\mathbb{C})$ such that $\iota(\alpha) = g \alpha g^{-1}$ for all $\alpha \in A\Gamma$.
- Thus we see that $\Gamma^{(2)} \subset g^{-1} \iota(O^1) g$ and has finite index.
- We conclude that $\Gamma$ and $\iota(O^1)$ are commensurable.
The Real Definition

We can now give the actual definition of arithmeticity and hopefully see how it is the same as our naive definition.

- Let $k$ be a number field with exactly 1 complex place and let $A$ be a quaternion algebra over $k$ which is ramified at all real places. Let $\iota$ be a $k$-embedding of $A$ into $M_2(\mathbb{C})$ and let $\mathcal{O}$ be an order of $A$. Then a subgroup $\Gamma$ of $\text{PSL}_2(\mathbb{C})$ is an arithmetic Kleinian group if it is commensurable with some $\iota(\mathcal{O})$.

Remark
In this case, the field, $k$, ends up being $k\Gamma$ and the algebra, $A$, turns out to be $A\Gamma$. 
Let $d < 0$, $k = \mathbb{Q}(\sqrt{d})$, and $\Gamma_d = \text{PSL}_2(\mathcal{O}_k)$.

- $k\Gamma_d = k$, which has exactly one complex place.
- $A\Gamma = M_2(k\Gamma_d)$, which is vacuously ramified at all real places of $k\Gamma$.
- $\mathcal{O} = M_2(\mathcal{O}_k)$.
- $\mathcal{O}^1 = \text{PSL}_2(\mathcal{O}_k)$. 
Why Should I Care About All This Arithmetic

You can learn about topology by looking at arithmetic

- Let $\Gamma$ have finite covolume. If $\Gamma$ is not cocompact then $A\Gamma = M_2(k\Gamma)$. If $\Gamma$ is arithmetic then the converse is also true, and in fact $\Gamma$ is commensurable with a Bianchi group.
- (Bass) If there exists $\gamma \in \Gamma$ such that $tr(\gamma)$ is not an algebraic integer, then $\mathbb{H}^3/\Gamma$ is Haken.
- (Lackenby) If $\Gamma$ is arithmetic and torsion free, then $\mathbb{H}^3/\Gamma$ contains an immersed $\pi_1$ injective surface.
- (Margulis) If $\Gamma$ is not arithmetic then the commensurability class of $\Gamma$ contains a unique maximal group.

If you liked this talk please thank Alan Reid.
If you did not like this talk please feel free to blame Eric Staron.